8.3 More About Zero Modes

8.3.1 Pure and Mixed Zero Modes

The JR zero mode constructs an operator of the form

$$\gamma = \sum_i \int dx \; f_i(x)\, c_i(x)$$

(In the presence of a soliton background!)

that satisfies $[H, \gamma] = 0$. (This gives the defining equation.)

i) Clearly $\gamma^2 = 0$.

ii) γ changes fermion number by one unit $([Q, \gamma] = -\gamma)$

iii) γ^+ is distinct from γ, and does not anticommute with it.

iv) We can normalize the f_i by requiring $\{\gamma, \gamma^+\} = 1$.

This gives $\sum_i \int dx\, |f_i(x)|^2 = 1$

The $p_x + i p_y$ zero mode constructs an operator of the form

$$\gamma = \sum_i \int dx \; f_i(x)\, c_i(x) + f_i^*(x)\, c_i^\dagger(x)$$

that satisfies $[H, \gamma] = 0$ in the presence of a vortex.

i) $\gamma = \gamma^\dagger$

ii) We can normalize γ by requiring $\gamma^2 = \frac{1}{2}\{\gamma, \gamma\} = 1$.

This gives $\sum_i \int dx \, |f_i(x)|^2 = 1$.

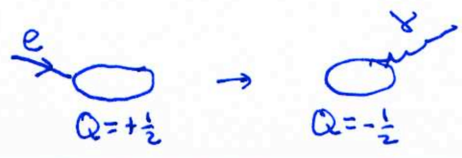
8.3.2 Physical Interpretation

In the JR case, one has degenerate states of the quantized soliton with $Q = \pm\frac{1}{2}$, depending on whether the 0-mode is occupied or not.

Allowed processes:

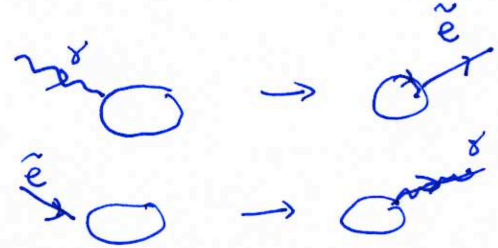
with $E_\gamma = E_e$

[N.B. : fixed background $\Leftrightarrow$ infinitely massive soliton; energy conserved but not momentum]



In the $p_x + i p_y$ case, charge (that is, fermion #) is conserved only modulo 2.

Allowed processes:



with $E_\gamma = E_{\tilde{e}}$

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Since $[H, \gamma] = 0$, at a formal level one can makes superpositions with $\frac{1 \pm \gamma}{2}$ (or $\frac{1 \pm (\gamma + \gamma^\dagger)}{2}$, $\frac{1 \pm \frac{\gamma - \gamma^\dagger}{i}}{2}$) projection operators, ... However the states are of very different character and one must worry about dephasing. [The relative phase between $|s\rangle$ + $\gamma |s\rangle$ does not grow linearly with time, but it could fluctuate due to interactions.]. In a cold system with a gap $\Delta \gg T$ it may be O.K. – and this is what we'll be exploiting for quantum statistics + topological computation.

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{Curiosity (?): Rotating $\frac{1+\gamma}{2}$ by 360° gives $\frac{1-\gamma}{2}$}

8.3.3 ~~Com~~ The Zero of Energy

[This question was raised by several of you.]

Q: Since the zero-point of energy is arbitrary - at least for non-gravitational interactions - what's special, or even meaningful, about 0 modes?

A: The operational significance is that they generate quasiparticle operators γ that satisfy $[H, \gamma] = 0$. These

[a constant here doesn't matter!]

yield degenerate states, and the physical processes sketched above (among others).

From the wave-equation point of view, the ~~operat~~ special choice of H is so that the spectrum is particle-hole (or C) symmetric.

0-mode is a mid-gap state.

8.3.4 Signatures

In principle one can work out the effective Hamiltonian in terms of quasiparticle operators. By including γ as one basis element in expanding ψ, and taking matrix elements, one can work out the special implications + signatures of zero-modes.

Some possibilities:

i) Specific heat at low T

ii) Residual entropy

iii) Hopping transport (?)

iv) In suitable cases, NMR

v) Split + study spectroscopy [specifically, in JR case, γ_5-Φ perturbation changes E in predictable way - also charge flow!]

This could be a very fruitful domain to investigate.