8.3 More About Zero Modes

8.3.1 Pure and Mixed Zero Modes

The JR zero mode constructs an operator of the form

$$8 = \sum_i dx \; F_i(x) \; c_i(x)$$

(In the presence of a soliton background.)

That satisfies $[H, 8] = 0$. (This gives the defining equation.)

i) Clearly $8^2 = 0$.

ii) $8$ changes fermion number by one unit. ($[Q, 8] = -8$)

iii) $8^\dagger$ is distinct from $8$, and does not anticommute with it.

iv) We can normalize the $f_i$ by requiring $\{8, 8^\dagger\} = 1$.

This gives

$$\sum_i \int dx \; |f_i(x)|^2 = 1$$
The $\psi\times\psi^\dagger$ zero mode constructs an operator of the form

$$\psi = \frac{1}{2} \int dx \, f_i(x) \, \psi_i(x) + f_i^*(x) \, \psi_i^\dagger(x)$$

that satisfies $[H, \psi] = 0$ in the presence of a vortex.

i) $\psi = \psi^*$

ii) We can normalize $\psi$ by requiring $\langle\psi | \psi\rangle = 1$.

This gives

$$\frac{1}{2} \int dx \, |f_i(x)|^2 = 1.$$

8.3.2 Physical Interpretation

In the JR case, one has states of the quantized soliton with $Q = \pm \frac{1}{2}$, depending on whether the 0-mode is occupied or not.

Allowed processes:

\[\begin{array}{c}
\text{e}^+ & \rightarrow & \text{e}^- \\
Q = +\frac{1}{2} & \Rightarrow & Q = -\frac{1}{2}
\end{array}\]

with $E_e = E_{e^+}$

[N.B.: fixed background $\Rightarrow$ infinitely massive soliton; every conserved but not momentum]
In the puripiy case, charge (that is, fermion #) is conserved only modulo 2.

Allowed processes:

\[ \begin{align*}
\pi^+ & \to e^+ \\
e^- & \to e^- 
\end{align*} \quad \text{with } E_e = E_{\pi^+}
\]

Since \([H, \pi^+] = 0\), at a formal level one can make superpositions with \(\frac{1 \pm \pi^+}{2}\) (or \(\frac{1 \pm \pi^+)^*}{2}\), \(\frac{1 \pm \pi^-}{2}\) projection operators, ... However the states are of very different character and one must worry about dephasing. [The relative phase between \(|\psi\rangle + |\psi \rangle\) does not grow with time, but it could fluctuate due to interactions.] In a cold system with a gap \(\Delta > T\) it may be O.K. — and this is what we'll be exploiting for quantum statistics + topological computation.

\[ \{ \text{Curiosity (2): Rotating } \frac{1}{2} \text{ by } 360^\circ \text{ gives } \frac{1}{2} \} \]
8.3.3 The Zero of Energy

[This question was raised by several of you.]

Q: Since the zero-point of energy is arbitrary— at least for non-gravitational interactions—what's special, or even meaningful, about 0 nodes?

A: The operational significance is that they generate quasiparticle operators \( \hat{\mathcal{G}} \) that satisfy \( [H, \hat{\mathcal{G}}] = 0 \). These [a constant here doesn't matter]
yield degenerate states, and the physical processes sketched above (among others).

From the wave-equation point of view, the choice of \( H \) is so that the spectrum is particle-hole (or C) symmetric.

\[ 0 \text{-node in a mid-gap state.} \]
8.3.4 Signatures

In principle one can work out the effective Hamiltonian in terms of quasiparticle operators. By including \$G\$ as one basis element in expanding it, and taking matrix elements, one can work out the special implications + signatures of zero-modes.

Some possibilities:

i) Specific heat at low \(T\)

ii) Residual entropy

iii) Hopping transport

iv) In suitable cases, NMR

v) Split + split, spectroscopy

This could be a very fruitful domain to investigate.