Lecture 9: Multiple Zero Modes, Braiding and Nonabelian Statistics

9.1 Philosophy

The 0-modes of Majorana (e.g. $p_x + ip_y$) type are localized structures of a well-characterized sort. Up to exponentially small corrections, they should be 0 energy states of high entanglement, as we encountered in the Kitaev model. Their charge-vortex structure gives us a way of transmitting phase information by global motions of separated objects, as we've seen both abstractly (AB effect, ~~anyon~~ classic anyons) and in the quantum Hall effect context [Lecture 10!]. So we get a huge Hilbert space that we can navigate using quasi-macroscopic handles, by having a large number of well-separated vortices. Motion in physical space $\mapsto$ motion in Hilbert space.

N.B. : i) This is not the only conceivable form of nonabelian statistics

ii) It is the form that occurs in Kitaev's model 2N, and predicted for $\nu = 5/2$ QHE.

9.2 Clifford Algebra

9.2.1 The algebra formed by Majorana-mode γ_i operators is one that is of great mathematical + physical importance - the Clifford algebra. Let's review it

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij} \qquad i = 1, \dots, N$$

Realized by

$$\gamma_1 = \sigma_1 \otimes 1 \otimes 1 \otimes 1 \cdots$$
$$\gamma_2 = \sigma_2 \otimes 1 \otimes 1 \otimes 1 \cdots$$
$$\gamma_3 = \sigma_3 \otimes \sigma_1 \otimes 1 \otimes 1 \cdots$$
$$\gamma_4 = \sigma_3 \otimes \sigma_2 \otimes 1 \otimes 1 \cdots$$
$$\gamma_5 = \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes 1 \otimes \cdots$$
$$\gamma_6 = \sigma_3 \otimes \sigma_3 \otimes \sigma_2 \otimes 1 \otimes \cdots$$

$\cdots$

9.2.1 The commutators $\frac{i}{4}[\gamma_i, \gamma_j] = L_{ij}$ satisfy the Lie algebra of $SO(N)$. So we get a representation of the group. This is basically the spinor representation. To analyze it more carefully we recognize that, for even N, $\Gamma = \gamma_1 \cdots \gamma_N$ commutes with all the γ_i, so we can project. For odd N, we can work in a smaller space by dropping the last factor ---(9.5.1) After these operations, the representation* is irreducible.

For even $N = 2n$, odd $N = 2n-1$ dimension of irreducible representation is 2^{n-1}.

9.2.2 The $c_i \equiv \frac{\gamma_{2i-1} + i\gamma_{2i}}{2}$, $c_i^+ \equiv \frac{\gamma_{2i-1} - i\gamma_{2i}}{2}$ satisfy the fermion algebra $\{c_i, c_j\} = \{c_i^+, c_j^+\} = 0$, $\{c_i, c_j^+\} = \delta_{ij}$

The conserving bilinears $c_i^+ c_j - c_j^+ c_i$ generate $SU(n)$...

9.2.3 The γ_i form a vector of $SO(N)$; the c_i form a vector of $SU(n)$.

9.2.4 The spins up or down in each component can be used to

* Another refinement: the different projections are inequivalent.

Yet another: Majorana ...

label the states (~~complete~~ maximal set of commuting operators $= L_{12}, L_{34}, \ldots$). This gives the "shift register"

$$|++--+\rangle$$
$$|+-++-\rangle$$
$$|---+-\rangle$$
$$\vdots$$

9.3 Braiding Operation and Nonabelian Statistics

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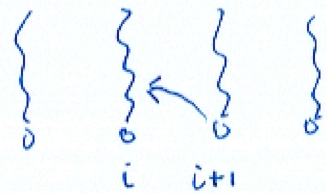
Taking a fermion (electron or hole) around an $\frac{h}{2e}$ vortex generates a $-$ sign. So we can implement interchange of vortex i with the neighboring vortex $i+1$ using

$$T_i: \quad \begin{aligned} &\gamma_i \mapsto \gamma_{i+1} \\ &\gamma_{i+1} \to -\gamma_i \\ &\gamma_j \mapsto \gamma_j \text{ otherwise} \end{aligned}$$

[Implicit here is a system of cuts for bookkeeping



Vertex i+1, with its 0 modes passes through the cut for vertex i]

One checks that these T_i satisfy the braid relations

$$T_i T_j = T_j T_i \qquad |i-j|>1$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \qquad \text{[exercise!]}$$

~~Up to an phase factor, we have can represent~~

We can represent the T_i by the unitary operators

$$\tau(T_i) = e^{\frac{\pi}{4}\gamma_{i+1}\gamma_i} = \frac{1}{\sqrt{2}}(1+\gamma_{i+1}\gamma_i)$$

– that is, $\quad \tau(T_i)\gamma_j\tau(T_i)^{-1} = T_i(c_j)$

This gives the non-abelian representation.
Basically, interchange of $i \leftrightarrow i+1$ in physical space gives rotation through $\frac{\pi}{2}$ in $(i,i+1)$ plane in spinor rep. in Hilbert space!

9.4 Examples

2 vertices

make complex fermion $c_1 = \frac{\gamma_1 + i\gamma_2}{2}$ $\quad c_1^+ = \frac{\gamma_1 - i\gamma_2}{2}$

$$\tau(T) = e^{\frac{\pi}{4}\gamma_2\gamma_1} = e^{i\frac{\pi}{4}(2c_1^+c_1 - 1)} = e^{i\frac{\pi}{4}\sigma_z}$$

Abelian so far, of course $(SO(2))$

4 vertices

$$\tau(T_1) = \exp\left(i\frac{\pi}{4}\sigma_z^{(1)}\right) = \begin{array}{cc} \begin{pmatrix} e^{-i\pi/4} & & & \\ & e^{i\pi/4} & & \\ & & e^{-i\pi/4} & \\ & & & e^{i\pi/4} \end{pmatrix} & \begin{array}{c} -\,- \\ +\,- \\ -\,+ \\ +\,+ \end{array} \end{array}$$

$$\tau(T_3) = \exp\left(i\frac{\pi}{4}\sigma_z^{(2)}\right) = \begin{pmatrix} e^{-i\pi/4} & & & \\ & e^{-i\pi/4} & & \\ & & e^{i\pi/4} & \\ & & & e^{i\pi/4} \end{pmatrix}$$

$$\tau(T_2) = \exp\frac{\pi}{4}\gamma_3\gamma_2 = \frac{1}{\sqrt{2}}(1+\gamma_3\gamma_2) = \frac{1}{\sqrt{2}}\left[1+i(c_2^{+}+c_2)(c_1^{+}-c_1)\right]$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix}.$$

Exercise: check this algebra, and find the operators for interchanges $1 \leftrightarrow 3$, $1 \leftrightarrow 4$, $2 \leftrightarrow 4$