9.1 Philosophy

The 0-modes of Majorana (e.g. \( p_+ \) or \( p_x \)) type are localized structures of a well-characterized sort. Up to exponentially small corrections, they should be low energy states of high entanglement, as we encountered in the Kitaev model.

Their charge-vortex structure gives us a way of transmitting phase information by global motion of separated objects, as we've seen both abstractly (AB effect, Cooper charge anyons) and in the quantum Hall effect context [Lecture 10.1]. So we get a huge Hilbert space that we can navigate using quasi-microscopic handles, by having a large number of well-separated vortices. Motion in physical space \( \to \) motion in Hilbert space.

N.B.: (i) This is not the only conceivable form of nonabelian statistics.
    (ii) It is the form that occurs in Kitaev's model 2N, and predicted for \( V = 5\% \), CTE.
9.2 Clifford Algebra

9.2.1 The algebra formed by Majorana-mode $\beta_i$ operators is one that is of great mathematical + physical importance - the Clifford algebra. Let's review it

$$\{\beta_i, \beta_j\} = 2 \delta_{ij} \quad i = 1, \ldots, N$$

Realized by

$$\begin{align*}
\beta_1 &= \sigma^1 \otimes 1 \otimes \cdots \\
\beta_2 &= \sigma^2 \otimes 1 \otimes \cdots \\
\beta_3 &= \sigma^3 \otimes 1 \otimes \cdots \\
\beta_4 &= \sigma^2 \otimes \sigma^3 \otimes \cdots \\
\beta_5 &= \sigma^3 \otimes \sigma^2 \otimes \cdots \\
\beta_6 &= \sigma^2 \otimes \sigma^3 \otimes \sigma \otimes \cdots \\
\end{align*}$$
The commutators \( \frac{i}{4} [x_i, x_j] = \delta_{ij} \) satisfy the Lie algebra of \( \text{SO}(N) \). So we get a representation of the group. This is basically the spinor representation. To analyze it more carefully we recognize that, for even \( N \), \( T = 8, \ldots, 8_N \) commute with all the \( x_i \), so we can project. For odd \( N \), we can work in a smaller space by dropping the last factor. After these operations, the representation is irreducible.

For even \( N = 2k \), the dimension of irreducible representation is \( 2^{n-1} \).

For odd \( N = 2k+1 \).

9.2.2 The \( x_i \Rightarrow x_{2i-1} + i x_{2i} \), \( x_i^+ \Rightarrow x_{2i-1} - i x_{2i} \) satisfy the fermion algebra \( \{ x_i, x_j \} = \{ x_i^+, x_j^+ \} = 0 \), \( \{ x_i, x_j^+ \} = \delta_{ij} \).

The conserving bilinears \( x_i^+ x_j - x_j^+ x_i \) generate \( \text{SU}(N) \).

9.2.3 The \( x_i \) form a vector of \( \text{SO}(N) \); the \( x_i \) form a vector of \( \text{SU}(N) \).

9.2.4 The spins up a down in each component can be used to...

Another refinement: the different projections are inequivalent. Yet another: Majorana...
9.3 Braiding Operation and Nonabelian Statistics

Taking a fermion (electron or hole) around an $\frac{1}{2} e$ vortex generates a $-\text{sign}$. So we can implement interchange of vortex $i$ with the neighboring vortex $i+1$.

Using

$$ T_i : \begin{align*}
    x_i &\rightarrow x_{i+1} \\
    x_{i+1} &\rightarrow -x_i \\
    x_j &\rightarrow x_j \text{ otherwise}
\end{align*} $$
Explicit here is a system of cuts for bookkeeping

\[
\begin{align*}
\{ & \{ \} \\
\{ & \{ \} \}
\end{align*}
\]

vertex \( i \), with its \( 0 \) modes, passes through the cut for vertex \( i \)

one checks that these \( T_i \) satisfy the braid relations

\[
T_i T_j = T_j T_i \quad |i-j| > 1
\]

\[
T_i T_{\text{in}} T_i = T_{\text{in}} T_i T_{\text{in}} \quad \text{[exercise!]} \]

Up to an exact phase factor, one can represent

we can represent the \( T_i \) by the unitary operators:

\[
\tau(T_i) = e^{i \xi_i \phi_i} = \frac{1}{2} \left( 1 + \xi_i \phi_i \right)
\]

that is,

\[
\tau(T_i) \xi_j \tau(T_i)^{-1} = T_i (c_j)
\]
This gives the non-abelian representation.

Basically, interchange of $i=\pm 1$ in physical space
gives rotation through $\frac{\pi}{2}$ in $(\mathbb{C},\mathbb{C})$ plane in spinor rep. in
Hilbert space!

9.4 Examples

2 vertices

\[ c_i = \frac{8_i + i\delta_2}{2}, \quad c_i' = \frac{8_i - i\delta_2}{2} \]

\[ \tau(T) = e^{\frac{\pi}{4} i\delta_2} = e^{i\frac{\pi}{4}} (2c_i c_i' - 1) = e^{i\frac{\pi}{8} \delta_2} \]

Abelian so far, of course (5.6.12)

4 vertices

\[ \tau(T_1) = \exp\left( i\frac{\pi}{4} \sigma_2 \right) = \begin{pmatrix} e^{-i\pi/4} & e^{i\pi/4} \\ e^{i\pi/4} & e^{-i\pi/4} \end{pmatrix} \]

\[ \tau(T_3) = \exp\left( i\frac{\pi}{4} \sigma_2 \right) = \begin{pmatrix} e^{-i\pi/4} & e^{i\pi/4} \\ e^{i\pi/4} & e^{-i\pi/4} \end{pmatrix} \]
\[ \tau(T_2) = \exp \frac{\pi}{4} \sigma_3 \sigma_2 = \frac{1}{\sqrt{2}} \left( 1 + \sigma_3 \sigma_2 \right) = \frac{1}{\sqrt{2}} \left[ 1 + i(c_2^* c_2)(c_1^* c_1) \right] \]

\[
= \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 0 & -i \\
0 & 1-i & 0 & 0 \\
0 & 0 & 1-i & 0 \\
-i & 0 & 0 & 1
\end{pmatrix}
\]

Exercise: check this algebra, and find the operators for interchanges 1 \rightarrow 3, 1 \rightarrow 4, 2 \rightarrow 4