9.4 Majorana Modes, State Counting and Locality

9.4.1 Realizing Algebras

The algebra of a single Majorana mode is $\gamma_1^2 = 1$, $[H, \gamma_1] = 0$. It can be realized on a single state, with $\gamma_1 |s\rangle = \pm |s\rangle$ ← choose one.

However that state cannot be invariant under 2π rotations, since 2π rotations change the sign of γ. [Remember, γ is linear in fermion creation/destruction operators.] So if we include 2π rotations as a symmetry, we need two states.

The algebra of two Majorana modes is $\gamma_1^2 = \gamma_2^2 = 1$, $\gamma_1\gamma_2 = -\gamma_2\gamma_1$, $[H, \gamma_{1,2}] = 0$. It can be realized on two states in a standard fashion, constructing fermion c/d operators

$$c_1^\dagger = \frac{\gamma_1 + i\gamma_2}{2}, \quad c = \frac{\gamma_1 - i\gamma_2}{2}, \quad \{c^\dagger, c\} = 1, \quad c^2 = c^{\dagger 2} = 0$$

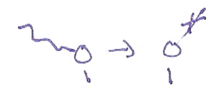
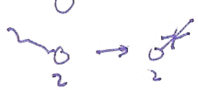
$$\begin{matrix} c|0\rangle = 0 & c^\dagger|0\rangle = |1\rangle \\ c|1\rangle = |0\rangle & c^\dagger|1\rangle = 0 \end{matrix} \quad \Rightarrow \quad \begin{matrix} \gamma_1|0\rangle = |1\rangle & \gamma_2|0\rangle = i|1\rangle \\ \gamma_1|1\rangle = |0\rangle & \gamma_2|1\rangle = -i|0\rangle \end{matrix} \qquad \begin{matrix} \gamma_1 \to \sigma_1 \\ \gamma_2 \to \sigma_2 \end{matrix}$$

This also involves two states. That's pretty startling, because our vortices might be far separated, so one might have expected a tensor product: $2 \times 2 = 4$. The "loss" of states directly reflects that γ_1 and γ_2 can't both be used to label, because they anticommute – even though they are localized on spacelike separated regions!

9.4.2 Physical Interpretation; Questions/Projects

We "see" that γ_1 (or γ_2) acted if a process

(or $1 \to 2$) with $\Delta E = 0$ occurs.

These thought-experiments would seem to generate 4 states for us (either do or don't implement each one). The way out is that a pair can go into or out of the condensate, so that  can make the same result as . And indeed γ_1 acts like γ_2, interchanging $|0\rangle \leftrightarrow |1\rangle$, up to phase factors.

Although this is logically impeccable, it would be good to see it explicitly. That is a very worthwhile project. Other questions:

For a single vortex in a finite sample, γ changes fermion # by 1. Is there still a near-zero mode, or does the parity of fermion # pick out one state of occupancy?

What happens when a vortex is transported from one edge to another (resistive flow)? Does it deliver a fermion to the edge?

Is there a significant amplitude for

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