9.5 Majorana Modes in a Very Simple Model

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9.5.1 The Model: A Quantum Wire

In 1 space dimension, spinless fermions on lattice points $j = 1, \ldots, L$.

$$H_1 = \sum_j -w\,(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) - \mu\,(c_j^\dagger c_j - \tfrac{1}{2}) + \Delta c_j c_{j+1} + \underbrace{\Delta^* c_{j+1}^\dagger c_j}$$

hopping — chemical potential — "condensate" interaction

$$\Delta \equiv |\Delta|\, e^{i\theta}$$

Write – introducing Majorana operators –

$$\gamma_{2j-1} = e^{i\frac{\theta}{2}} c_j + e^{-i\frac{\theta}{2}} c_j^\dagger \qquad \gamma_{2j} = -i\,(e^{i\frac{\theta}{2}} c_j - e^{-i\frac{\theta}{2}} c_j^\dagger)$$

These satisfy the Clifford algebra $\{\gamma_l, \gamma_m\} = 2\delta_{lm}$ and $\gamma_l^* = \gamma_l$. Now

$$H_1 = \frac{i}{2} \sum_j \Big[-\mu\, \gamma_{2j-1}\gamma_{2j} + (w + |\Delta|)\,\gamma_{2j}\gamma_{2j+1} + (-w + |\Delta|)\,\gamma_{2j-1}\gamma_{2j+2}$$

so $\quad c_j = e^{-i\frac{\theta}{2}}\left(\dfrac{\gamma_{2j-1} + i\gamma_{2j}}{2}\right) \qquad c_j^\dagger = e^{i\frac{\theta}{2}}\left(\dfrac{\gamma_{2j-1} - i\gamma_{2j}}{2}\right)$

9.5.2 Two simple phases

$$\boxed{H_1 = \frac{i}{2}\sum_j \left[-\mu \gamma_{2j-1}\gamma_{2j} + (w+|\Delta|)\gamma_{2j}\gamma_{2j+1} + (-w+|\Delta|)\gamma_{2j-1}\gamma_{2j+2} \right]}$$

a) If $|\Delta| = w = 0$, $\mu < 0$ (totally trivial!)

$$H_1 = \sum -\mu \left(c_j^\dagger c_j - \tfrac{1}{2} \right)$$
$$= -\frac{i}{2}\mu \sum \gamma_{2j-1}\gamma_{2j}$$

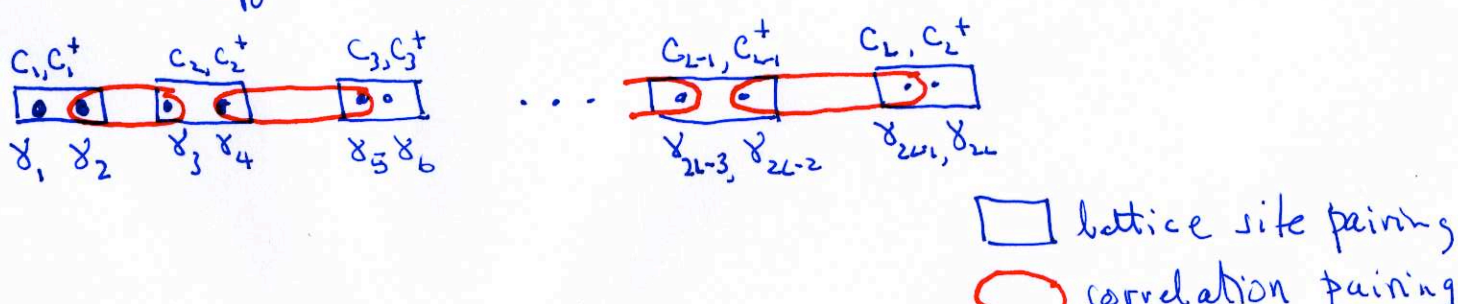
ground state: totally unoccupied

the γ_{2j-1} and γ_{2j} pair off $\gamma_{2j-1}\gamma_{2j}|g\rangle = -i|g\rangle$

b) If $|\Delta| = w > 0$, $\mu = 0$

$$H_1 = \frac{i}{2} w \sum \gamma_{2j}\gamma_{2j+1}$$

so γ_{2j} pairs with γ_{2j+1}. Note that these came from different sites!



$$\gamma_{2j}\gamma_{2j+1}|g\rangle = i|g\rangle$$

γ_1 and γ_2 decouple from the Hamiltonian!

We have 2 ~~degenerate~~ degenerate ground states $|\psi_0\rangle$, $|\psi_1\rangle$ with

$$\gamma_1\gamma_{2L}|\psi_0\rangle = i|\psi_0\rangle \qquad \gamma_1\gamma_{2L}|\psi_1\rangle = -i|\psi_1\rangle$$

[Note: We are simultaneously diagonalizing commuting operators ...]

The F-parity operator

$$P = \prod_j (1-2c_j^+c_j) = \prod_j (-i\gamma_{2j-1}\gamma_{2j})$$

commutes with all our bilinears. We see

$$P|\psi_0\rangle = |\psi_0\rangle \qquad P|\psi_1\rangle = -|\psi_1\rangle$$

so $|\psi_0\rangle$ contains ~~and~~ an even # of fermions, $|\psi_1\rangle$ an odd number.

9.5.3 Pumping

An adiabatic change in the phase of the order parameter θ through 2π changes $\gamma_i \to -\gamma_i$.

[Exercise: Make this a good argument, using Berry's phase]

This same transformation, $\gamma_i \to -\gamma_i$, is implemented by P, regarded as a unitary transformation (Pulling γ_i through gives an odd # of - signs: one for every index 1,...,2L except i.)

This same transformation is implemented by passing an electron from one side to the other!

Interpretation: We go to the same ground state, but an electron is transported from one end to the other.

[Project: really demonstrate that!]
- follow the flow