9.5 Majorana Modes in a Very Simple Model

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9.5.1 The Model: A Quantum Wire

In 1 space dimension, spinless fermions on lattice points $j = 1, \ldots, L$.

$$H = \sum_j -\omega (c_j^+ c_{j+1} + c_{j+1}^+ c_j) - \mu (c_j^+ c_j - \frac{1}{2}) + \Delta c_j c_{j+1} + \Delta^* c_j^+ c_{j+1}$$

*chemical potential*

$\Delta = |\Delta| e^{i\theta}$

*"condensate" interaction*

Write - introducing Majorana operators -

$$\gamma_{2j-1} = e^{i\frac{\theta}{2}} c_j + e^{-i\frac{\theta}{2}} c_j^+$$

$$\gamma_{2j} = -i (e^{i\frac{\theta}{2}} c_j - e^{-i\frac{\theta}{2}} c_j^+)$$

These satisfy the Clifford algebra $[\gamma_k, \gamma_l] = 2\delta_{kl}$ and $\gamma_k^* = \gamma_k$. Now

$$H = \frac{i}{2} \sum_j \left[-\mu \gamma_{2j-1} \gamma_{2j} + (\omega + 1i\Delta) \gamma_{2j} \gamma_{2j+1} + (-\omega + 1i\Delta) \gamma_{2j-1} \gamma_{2j+2}\right]$$

So

$$c_j = e^{-i\frac{\theta}{2}} \left( \frac{\gamma_{2j-1} + i \gamma_{2j}}{2} \right)$$

$$c_j^+ = e^{i\frac{\theta}{2}} \left( \frac{\gamma_{2j-1} - i \gamma_{2j}}{2} \right)$$
9.5.2 Two simple phases

a) If $|\Delta|=\omega=0$, $\mu<0$ (totally trivial)

$$H_1 = \sum_j -\mu(x_j^+ x_j - \frac{1}{2})$$

$$= -\frac{i}{2} \mu \sum_j x_{2j-1} x_{2j}$$

Ground state: totally unoccupied

The $x_{2j-1}$ and $x_{2j}$ pair off

$$|x_{2j-1} x_{2j} \rangle = -i |x \rangle$$

b) If $|\Delta|=\omega>0$, $\mu=0$

$$H_1 = \frac{i}{2} \omega \sum_j x_{2j}^+ x_{2j+1}$$

So $x_{2j}$ pair, with $x_{2j+1}$. Note that these care for different sites!

$$|x_{2j} x_{2j+1} \rangle = i |x \rangle$$

\[\begin{array}{cccc}
C_1, C_1^+ & C_2, C_2^+ & C_3, C_3^+ & \ldots \\
\circ & \bullet & \bullet & \ldots \\
\end{array}\]

\[\begin{array}{cccc}
C_{L-1}, C_{L-1}^+ & C_L, C_L^+ & C_{L+1}, C_{L+1}^+ & \ldots \\
\circ & \circ & \bullet & \ldots \\
\end{array}\]

\[\begin{array}{cccc}
x_1, x_2, x_3, x_4 & x_5, x_6 & x_{2L-3}, x_{2L-2}, x_{2L-1}, x_{2L} & \ldots \\
\bullet & \circ & \circ & \ldots \\
\end{array}\]

\[\begin{array}{cccc}
x_{2j}, x_{2j+1} & \ldots & \ldots & \ldots \\
\bullet & \circ & \circ & \ldots \\
\end{array}\]

\[\begin{array}{cccc}
lattice site pairing & \text{correlation pairing} & \text{correlation pairing} & \ldots \\
\circ & \circ & \circ & \ldots \\
\end{array}\]
\( y_i \) and \( y_j \) decouple from the Hamiltonian!

We have 2 degenerate ground states \( |\Psi_0\rangle, |\Psi_1\rangle \) with
\[
|\Psi_0\rangle = i |\Psi_0\rangle \quad |\Psi_1\rangle = -i |\Psi_1\rangle
\]

(Note: We are simultaneously diagonalizing commuting operators --)

The F-parity operator
\[
P = \prod_j (1 - 2c_j^* c_j) = \prod_j (-i \, y_{2j}, y_{2j})
\]
commutes with all our bilinears. We see
\[
P |\Psi_0\rangle = |\Psi_0\rangle \quad P |\Psi_1\rangle = - |\Psi_1\rangle
\]
so \(|\Psi_0\rangle\) contains an even # of fermions, \(|\Psi_1\rangle\) an odd number.

9.5.3 Pumping

An adiabatic change in the order parameter \( \Theta \) through

\( 2\pi \) changes \( y_i \to -y_i \).

[Exercise: Make this a good argument, using Berry's phase]
This same transformation, \( \chi_i \rightarrow -\chi_i \), is implemented by \( P \), regarded as a unitary transformation (Pulling \( \chi_i \) through gives an odd \# of - signs = one for every index \( i \), except \( i \)).

This same transformation is implemented by passing an electron from one side to the other!

Interpretation: We go to the same ground state, but an electron is transported from one end to the other.

[Prop: really demonstrate that!]

- follows the flow