Quantum Hall Effect I: Basic Theory

10.0 Phenomenon

Large B, 2d high mobility electron gas at low T.

Transverse (Hall) conductance exhibits plateaus at values related to fundamental constants: $\sigma_{xy} = \nu \frac{e^2}{h}$, ν, rational.

On these plateaus $\rho_{xx} = 0$ (no dissipation).

Interpretation (1st pass): At preferred densities ~~i~~ electrons form an incompressible quantum liquid + localized states pinned to impurities. The preferred densities are $n = \frac{\nu e B}{h}$, corresponding to

$N = \nu \frac{BA}{h/e}$. $\nu \equiv$ filling factor [Hall: $\sigma_{xy} = \frac{ne}{B}$]

Many ν have been observed. Almost all have odd denominators.

[Foregoing: double layers
spin]

10.1 Ideal Theory I: Integer

10.1.1 Landau levels

$$H = -\frac{1}{2m}\left(\partial_j - iqA_j\right)^2$$

$$(A_x, A_y) = \frac{B}{2}(-y, x)$$

[$\propto \hat{z}\times\vec{r}$
$\Rightarrow$ manifestly rotationally invariant
"symmetric gauge"]

Define

$$\partial_z = \frac{1}{2}\left(\partial_x - i\partial_y\right)$$

$$\partial_{\bar z} = \frac{1}{2}\left(\partial_x + i\partial_y\right)$$

$$A_z = \frac{1}{2}\left(A_x - iA_y\right) = \frac{B}{4i}\bar{z}$$

$$A_{\bar z} = \frac{1}{2}\left(A_x + iA_y\right) = -\frac{B}{4i}z$$

[$z = x + iy$ ...
$\bar{z} = x - iy$]

Then

$$H = -\frac{1}{m}\left(D_z D_{\bar z} + D_{\bar z} D_z\right)$$

(with $D_z = \partial_z - iqA_z$ ...)

This can be simplified using a similarity transformation:

$$H = e^{-|z|^2/4\ell_B^2}\left[-\frac{1}{m}\left(\tilde{D}_z\tilde{D}_{\bar{z}} + \tilde{D}_{\bar{z}}\tilde{D}\right)\right]e^{+|z|^2/4\ell_B^2}$$

$$\ell_B^2 \equiv \left|\frac{1}{qB}\right| \;;\; \ell_B \equiv \text{magnetic length}$$

Assuming $qB > 0$,

$$\tilde{D}_z = e^{|z|^2/4\ell_B^2} D_z\, e^{-|z|^2/4\ell_B^2} = \partial_z - \frac{1}{2\ell_B^2}\bar{z}$$

$$\tilde{D}_{\bar{z}} = e^{|z|^2/4\ell_B^2} D_{\bar{z}}\, e^{-|z|^2/4\ell_B^2} = \partial_{\bar{z}}$$

(for $qB<0$; $\tilde{D}_z = \partial_z$ etc.)

$$H = e^{-|z|^2/4\ell_B^2}\left(-\frac{2}{m}\left(\partial_z - \frac{1}{2\ell_B^2}\bar{z}\right)\partial_{\bar{z}}\right)e^{+|z|^2/4\ell_B^2} + \underbrace{\frac{1}{2}\frac{qB}{m}}_{\equiv \frac{1}{2}\omega_c}$$

Eigenstates

Ψ_0: $\quad e^{-|z|^2/4\ell_B^2} f(z) \qquad E = \frac{1}{2}\omega_c$

N.B.: holomorphic

Ψ_1: $\quad e^{-|z|^2/4\ell_B^2}\left(\partial_z - \frac{1}{2\ell_B^2}\bar{z}\right) f(z) \qquad E = \left(\frac{1}{2}+1\right)\omega_c$

Ψ_n: $\quad e^{-|z|^2/4\ell_B^2}\left(\partial_z - \frac{1}{2\ell_B^2}\bar{z}\right)^n f(z) \qquad E = \left(\frac{1}{2}+n\right)\omega_c$

[In Landau gauge, plane wave × harmonic oscillator]

10.2 Droplet; flux forcing

The power of z ($-$ the power of $\bar{z}$) encodes the angular momentum.

For large l in the lowest Landau level

$$\psi_0^l \propto e^{-|z|^2/4l_B^2} z^l$$

is sharply peaked at $r_l = \sqrt{2 l_B^2 l}$. It makes an annulus of width $\sim l_B$. The flux enclosed inside r_l is $\pi r_l^2 B = 2\pi l_B^2 l B = \frac{2\pi}{q} l$, i.e. l flux quanta (N.B. $\hbar = 1$).

A droplet centered at the origin is described by

$$\Psi(z_1, \dots, z_N) = \text{Antisymmetrize}\left[z_1^0, z_2^1, \dots z_N^{N-1}\right] e^{-\frac{\sum |z_i|^2}{4l_B^2}}$$

$$= \det \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_N \\ & & \vdots & \\ z_1^{N-1} & & \cdots & z_N^{N-1} \end{vmatrix} e^{-\frac{\sum |z_i|^2}{4l_B^2}}$$

Van der Monde determinant

$$= \prod_{i<j} (z_i - z_j)\, e^{-\sum |z_i|^2/4 l_B^2}$$

(Proof: zeroes and degree)

10.1.3 Density; Plasma Analogy

The droplet is "clearly" of uniform density, since the underlying system is translation invariant, and all the states (rings) that occur within r_{max} are occupied. (There is a fall-off very near the edge, due to the finite width of the rings).

Another, more flexible heuristic is Laughlin's plasma analogy

$$|\Psi|^2 = e^{-\beta V(z_1,\ldots,z_N)}$$

$$V = -\sum_{i<j} \ln|z_i - z_j| + \frac{1}{4\ell_B^2}\sum |z_i|^2$$

repulsive log potential (2d)

uniform background charge (2d)

$$\beta = \frac{2}{m}$$

neutrality: $\pi R^2 \rho = 2\pi R E$; $E = \frac{\rho R}{2}$; $\phi = \frac{\rho R^2}{4}$

$$\Rightarrow \rho = \frac{1}{\ell_B^2} = qB \quad (\nu = 1)$$

If we adiabatically add a flux quantum at the origin, acquire an additional factor

$$\prod_{i=1}^{N} z_i$$

This agrees with Faraday. The intermediate states have factors $\prod z_i^{\alpha}$ – fractional angular momentum.

10.2 Ideal Theory II: Basic Fractions ($\nu = 1/m$)

10.2.1 Laughlin's trial wavefunction

$$\Psi(z_1, \dots, z_N) = \prod_{i<j} (z_i - z_j)^m \; e^{-\sum |z_i|^2/4}$$

m odd

i) It is antisymmetric (for m odd only).
"Trial" in Jastrow form, keeping electrons apart.

ii) Uniform density $\quad \nu q B = \frac{1}{m} q B$

argument 1: You get this wavefunction from the $\nu = 1$ droplet by raising to the m^{th} power and re-scaling z. So it inherits uniformity.

To get the density, count power of z: $1/m$ of states occupied!

argument 2: Plasma analogy. $\beta \to \beta \times m$, ← to keep unit charge

background charge → b.c. $\times \frac{1}{m}$.

~~argument 3~~:

iii) Exact groundstate with repulsive 2-body potential $\propto \partial_z^{m-1} \delta^2(z)$, which enforces high-order zeroes.

iv) (My favorite): Statistical transmutation

Small magnetic field is not a small perturbation, since $\vec{A}$ grows.

We can cancel it off (i.e., the nasty tail) by attaching an additional vector potential $\vec{a}$, of the type used to implement statistical transmutation.

If the "transmutation" takes fermions to fermions, we have an acceptable candidate state

Trading along lines

$$\Delta \frac{\theta}{\pi} = \Delta \frac{1}{\nu}$$

keeps the long-range potential unperturbed

$$\left[\begin{array}{l} \Delta \frac{\theta}{\pi} = \Delta \phi \text{ (fictitious flux)} \\ \Delta qBA = \Delta \phi \rho A \\ \text{[real flux]} \end{array} \Rightarrow \Delta \frac{\theta}{\pi} = \Delta \frac{qB}{\rho} = \Delta \frac{1}{\nu} \right]$$

The "residual" perturbation is the difference between a small uniform flux and a small flux (of the opposite total magnitude) localized to particles.

Generically, adiabatic evolution using such a perturbation will leave a gapped state gapped, and qualitatively similar.

From $\nu = 1$ to $\nu = 1/3$ (or $1/2m+1$)

requires $\Delta \frac{\theta}{\pi} = 2$ $(2m)$, so it takes

fermions $\to$ fermions

full L.L. $\to$ fractional L.L.

[Comment: two cancelling fictitious (i.e., in overall flux) fields $\overset{?}{\Rightarrow}$ repulsive bosons]

$|z_i - z_j|^2$

This set-up has been used to perturb in one step (poorly controlled) or from fermions $\to 1 - \frac{1}{n}$ anyons (anyon superconductivity).

Challenge: do more with this beautiful idea!

10.2.2 Elementary Excitations

"Move 'em out" without upsetting correlations by inserting flux

$$\psi_{z_0} = \prod (z_i - z_0) \prod (z_i - z_j)^3 \, e^{-\sum |z_i|^2/4\ell_B^2}$$

This produces a fractional charge $-q/3$ quasihole.

i) Doing it three times produces a vacancy ready for electron occupation.

ii) Plasma analogy; neutralizing charge $= q/3$

iii) A Berry phase argument, which also proves fractional statistics!

10.2.3 Berry Phase Technique

Hamiltonians depending on a parameter $\lambda(t)$ adiabatically. Non-degenerate states $\psi(t)$, satisfying

$$i\frac{\partial \psi(t)}{\partial t} = H(\lambda(t))\,\psi(t) = E(\lambda(t))\,\psi(t)$$

According to adiabatic theorem ("no quantum jumps")

$$\psi(t) \approx u(\lambda(t))\, e^{-i\int^t E(\lambda(u))\,du}\, e^{i\eta(t)}$$

$$H(\lambda(t))\,u(\lambda(t)) = E(\lambda(t))\,u(\lambda(t))$$

Plugging this in:

$$i\frac{\partial u}{\partial \lambda}\dot{\lambda} - \dot{\eta} u = 0$$

Hit with $(u, -)$:

$$\dot{\eta} = i\underbrace{\langle u | \frac{\partial u}{\partial \lambda}\rangle}_{\text{N.B.: imaginary}}\dot{\lambda}$$

$$d\eta = i\langle u|\frac{\partial u}{\partial \lambda}\rangle\, d\lambda$$

Going around a closed loop, we get a convention-independent phase that reflects the geometry of Hilbert space:

$$\oint d\eta = i\oint \langle u|\frac{\partial u}{\partial \lambda}\rangle\, d\lambda$$

$\langle u|\frac{\partial u}{\partial \lambda}\rangle$ can be considered as a gauge field on parameter space.

10.2.4 Application to Quasihole Charge and Statistics

We study this by notionally moving the quasihole around, and matching the "observed" behavior to a charged anyon.

$$d\alpha = i\left\langle \psi_{z_0} \middle| \frac{\partial \psi_{z_0}}{\partial z_0} \right\rangle = i\left\langle \psi_{z_0} \middle| \sum \frac{1}{z_i - z_0} \middle| \psi_{z_0} \right\rangle$$

Integrate around loop, using Cauchy

$$i \oint \left\langle \psi_{z_0} \middle| \frac{1}{z_i - z_0} \middle| \psi_{z_0} \right\rangle = \begin{array}{l} +2\pi \text{ for } z_i \text{ inside loop} \\ 0 \text{ for } z_i \text{ outside loop} \end{array}$$

Thus $\oint d\alpha = 2\pi \rho A$

A particle of charge q^* should acquire phase

$$q^* \Phi = q^* B A$$

Thus $q^* = \frac{2\pi\rho}{B} = \nu q$.

Once we know this, we see (using the same argument) that moving one quasihole around another gives an extra phase

$$2\pi \Delta\rho A = 2\pi \frac{q^*}{q} = 2\pi\nu$$

10.3: Dirty Theory

- 10.3.1 The ideal theory does not really explain the existence of plateaus over a continuous range of densities. There is a formal gap in energy, but it does not allow you to add states one by one. (You get a whole Landau level, or nothing.)
- 10.3.2 In the realistic situation, there is a mobility edge separating localized from delocalized states. When the chemical potential is in the localized region, you can add states one by one without affecting the conductivity.

- 10.3.3 The Laughlin/Halperin argument: Put a unit fluxoid through the center of an annulus of quantum Hall material connected to reservoirs (leads). All that can happen is that an integral number of electrons is transferred between the reservoirs. We - following Faraday and Landau - saw earlier that flux wants to do this.

$$\text{Work} = \int J \cdot E \, dt = \int J \frac{d\Phi}{dt} dt = \int J \, d\Phi = \bar{J} \frac{h}{e}$$

$$\shortparallel$$

$$ne\,\delta\mu = neV$$

# of electrons transferred

$$\text{so} \quad \bar{J}/V = \frac{ne^2}{h}$$