11.2.2 Ground state trial wavefunction for $\nu = \frac{1}{2}$

$$\chi_{\frac{1}{2}}(z_1, \ldots, z_N) = \text{Pf} \frac{1}{z_i - z_j} \prod (z_i - z_j)^2 e^{-\frac{1}{4\hbar^2} \sum z_i^2}$$

Pairing in Laughlin bosonic form

Pair up (i.e. $\nu = -1$)

Note: Antisymmetric
Non-singular

11.2.3 The filling fraction is in fact $\frac{1}{2}$, since the order of the polynomial is $2[N(N-1)]$ from the repulsive $(z_i - z_j)^2$ minus just $\frac{N}{2}$ from the Pfaffian.

Although this is written for the lowest Landau level only, it can be used with minor modifications for $\nu = \frac{3}{2}, \frac{5}{2}, \ldots$, where Landau levels be well below the Fermi energy are completely full and essentially inert.
11.2.4 There are several diagnostics for the proposed universality class, given numerical simulation of electrons subject to a given Hamiltonian. They include:

a) Presence of a gap, complete spin polarization
b) Discrete ground state degeneracies depending on topology, e.g., on sphere, torus, + higher genus surfaces.
This is similar in spirit to, but considerably more complicated than, the phenomenon we saw in the Kitaev model.
We'll analyze it later. Ref: cond-mat/0607743

c) N-dependent degeneracies for disc geometry (= edge states).
We'll analyze it later. Ref: cond-mat/0906453

Topological field theories can be used to calculate numbers like in b) + c). These are theories with no conventional local degrees of freedom, but global structure. Example: Discrete gauge theories; pure Chern-Simons theories

\[ \Rightarrow \text{eq. of motion: } f_{\mu \nu} = 0 \]
11.2.5 There is good experimental evidence for a gapped, effectively spin-polarized state at $\nu = 5/2$. Numerical simulations with interactions thought to be realistic for the experiment in question (i.e., screening, Coulomb,...) favor a Pfaffian ground state, according to the criteria just mentioned.

11.3 Quasiholes in the Pfaffian state

11.3.1 The quasihole in this universality class are predicted to have remarkable properties, including both abelian and non-abelian quantum statistics. The abelian part is much as before, with a twist we'll come to immediately; the

S + $\nu = 1/2 \pm 1$

non-abelian part is much as in 11.2.6. The flux-trading heuristic, we relate $\nu = \frac{1}{2}$ to

11.3.2 According to the flux-trading heuristic, we relate $\nu = \frac{1}{2}$ to fermions with a gap in zero magnetic field. We're interested in effectively spinless fermions, so the simplest pairing possibility is $p$-wave, and to have a gap we like $p$+i$p$
11.3.3 In this procedure the electrons acquire fictitious flux \( \Phi = \frac{2\hbar}{q} \). For a paired state the elementary flux should be
\[ \Phi = \frac{\hbar}{2q} \] Thus we might expect to have quasiparticles that are \( \frac{1}{4} \) of an electron (a hole), with charge \( e/4 \) and statistic \( \pi/4 \).

11.3.4 There is a subtlety in the statistics, since if 4 quasiparticles are interchanged with 4 others we get \( (e^{i\pi/4})^4 = 1 \) - bosons, not fermions!

[This works out for \( \frac{1}{m} \) states: \( (e^{i\pi/m})^m = e^{i\pi} = -1 \).]

\( m \text{ odd!} \)

{ Exactly this is repaired by the \( Z_4 \) zero-mode. } (?? conjecture)

\[ \text{Four quasiholes + electron make a neutral fermion, which must also appear in the spectrum.} \]
11.3.5 The wavefunction for quasiholes, that realizes all this intuition, is constructed as follows.

For a quasihole at \( z_0 \), modify the Pfaffian according to

\[
\text{Pf} \left( \frac{1}{z_i - z_j} \right) \rightarrow \text{Pf} \left( \frac{z_i - z_0}{z_i - z_j} \right)
\]

This gives \( \frac{1}{2} \) a Laughlin factor, and \( \frac{1}{4} \) of an electron factor.

It boosts the orbital angular momentum of each pair by 1 unit.

11.3.6 For 2 quasiholes at \( z_1, z_2 \), the appropriate modification (which reduces to a Laughlin flux as \( z_2 \to z_1 \)) is

\[
\text{Pf} \left( \frac{1}{z_i - z_j} \right) \rightarrow \text{Pf} \left( \frac{(z_i - z_1)(z_j - z_2)}{z_i - z_j} \right) + \frac{1}{2} (z_j - z_1)(z_i - z_2)
\]
For four quasiparticles there appear, at first glance, to be three distinct possibilities for the numerator in the Pfaffian:

\[(z_i - m_1)(z_i - m_2)(z_j - m_3)(z_j - m_4) + (i \leftrightarrow j) \quad \text{Write:} \quad (12)(34)\]

\[(z_i - m_1)(z_i - m_3)(z_j - m_2)(z_j - m_4) + (i \leftrightarrow j) \quad \equiv \quad (13)(24)\]

\[(z_i - m_1)(z_i - m_4)(z_j - m_2)(z_j - m_3) + (i \leftrightarrow j) \quad \equiv \quad (14)(23)\]

But note \((12)(34) - (13)(24) = (z_i - z_j)^2 (m_i - m_4)(m_2 - m_3)\)

This vanishes as \(z_i \rightarrow z_j\), symmetric, order-2

vanishes as \(m_i \rightarrow m_4\) or \(m_2 \rightarrow m_3\)

So

\[
\frac{(12)(34) - (13)(24)}{(12)(34) - (14)(23)} = \frac{(m_i - m_4)(m_2 - m_3)}{(m_i - m_3)(m_2 - m_4)}
\]

("cross-ratio")

is independent of \(z\)!

Thus there are only 2 independent configurations. - for two electrons!

(note: Pfaffian is non-linear)
For a larger number of electrons, with a little more work we prove that

\[ \text{Pf}_{(12)(34)} - \text{Pf}_{(14)(23)} = \frac{m_{14} m_{23}}{m_{13} m_{24}} \left( \text{Pf}_{(12)(34)} - \text{Pf}_{(13)(24)} \right) \]

\[ \Leftrightarrow \text{c-i.e.} \frac{(n_1-n_4)(n_2-n_3)}{(n_1-n_3)(n_2-n_4)} \]

In general, (c. Nayal + F. W., NP B479, 529)

11.3.10 The generalization to more quasiholes gives a basis as follows:

\[ \text{Pf}_{(m_1 \ldots n_1)(m_2 \ldots n_2)} \]

subject to the restriction that \( m_1, m_2 \) appear in one Latin and one Greek grouping, as do \( n_1, n_4; n_3, n_6 \), etc.

This gives \( 2^{n/2} = 2^{n-1} \) basis states, for \( 2n \) quasiholes

Interpretation: \( m_1, m_2 \) are "paired" in the sense that as \( m_1 \to m_2 \), we get a conventional (Laughlin) vortex.
A distant vortex does not resolve Cooper pairs. As we bring $\frac{1}{2}$-vortices in, two at a time, each electron in a pair must see one of them. The first choice is arbitrary, but subsequent ones build unambiguously on the previous:

$$
\begin{array}{c}
1 \quad 2 \\
\downarrow \\
13 \quad 24 \\
\text{or} \\
14 \quad 23 \\
\end{array}
\quad \rightarrow
\begin{array}{c}
135 \quad 246 \\
0_1 \quad 136 \quad 254 \\
\text{or} \\
0_1 \quad 145 \quad 236 \\
\end{array}
\text{etc.}

11.3.11 In principle one can now use the Berry phase to compute braiding, including the non-abelian factor. But this has never been done! [Project]

(I think it's very doable, though.)

Using conformal field theory, we showed the spinor representation emerged, as for Majorana modes. [Original insight: Moore & Read]

Now we'll just restate it.

It's instructive to approach the situation in different ways.
11.4 Relation to pairing superconductor

Let's recall the relevant essence of BCS theory.

\[ \text{Keff.} = \sum_{k} \varepsilon_k \, C_k^+ C_k + \frac{1}{2} (\Delta_k^* C_k C_k + \Delta_k C_k^+ C_{-k}^-) \]

\[ \varepsilon_k = \varepsilon_k - \mu = \frac{k^2}{2m} - \mu \]

\[ \Delta_k \sim \Delta(k_x - iky) \text{ for small } k \]

\[ \rightarrow 0 \text{ for } k \rightarrow \infty \]

\[ |\phi\rangle = \prod_{k} (u_k + u_k^* C_k^+ C_{-k}^-) |0\rangle \]

\[ \alpha_k = u_k C_k - u_k^* C_k^- \]
\[ \alpha_k^+ = u_k^* C_k^+ - u_k C_k \]

\[ \{ \alpha_k, \alpha_k^+ \} = \delta_{kk} ; \quad \alpha_k |\phi\rangle = 0 \]

Demanding \[ [\alpha_k, \text{Keff.}] = E_k \alpha_k \], we diagonalize \[ \text{Keff.} = \sum_{k} E_k \alpha_k^+ \alpha_k \]

\[ E_k > 0 \]
The wave equation (Bogoliubov - de Gennes) is

\[ E_k u_k = \varepsilon_k u_k - \Delta_k^* u_k \]
\[ E_k v_k = -\varepsilon_k v_k - \Delta_k u_k \]

which leads to

\[ E_k = \sqrt{\varepsilon_k^2 + 4 \Delta_k^2} \]
\[ u_k / u_k = - (E_k - \varepsilon_k) / \Delta_k^* \]
\[ |u_k|^2 = \frac{1}{2} \left( 1 + \frac{\varepsilon_k}{E_k} \right) = 1 - |v_k|^2 \]

11.4.2 The ground state can be written (up to a phase)

\[ \left| \Psi \right> = \prod (u_k + v_k c_k^+ c_{-k}^+) \left| 0 \right> = \prod |u_k|^{\frac{1}{2}} \exp \left( \frac{i}{2} \sum_k g_k c_k^+ c_{-k}^+ \right) \left| 0 \right> \]

with \( g_k = u_k / u_k \)

Expanding and passing to real space, we find in the \( N \) electron sector

\[ \Psi (r_1, ..., r_N) \propto \Sigma \text{sign} \prod_{i=1}^{N/2} \tilde{g} (r_i (2i-1) - r_i (2i-1)) \]

(note: \( N \) is even!)

\[ \text{Pfaffian!} \]
11.4.3 **Distinguish:**

**Strong pairing** ("molecular BEC"): \( \varepsilon_k > 0 \) as \( k \to 0 \)

\[ \Rightarrow |\psi_k| \to 1, |\varphi_k| \to 0 \]

unoccupied

**Weak pairing** (BCS - but not necessarily very weak coupling): \( \varepsilon_k < 0 \) as \( k \to 0 \)

\[ \Rightarrow |\psi_k| \to 0, |\varphi_k| \to 1 \]

In strong pairing, \( g_k = \psi_k / |\psi_k| \to \mu e^{iky} \) as \( k \to 0 \). It is well-behaved at \( \infty \), and analytic, \( g_k \sim e^{-\mu \rho} \).

In weak pairing, \( g_k \to \infty \frac{1}{k x - iky} \) as \( k \to 0 \).

\[ g(r) \sim \int \frac{dk}{k x - iky} \frac{e^{ikr}}{r} \to \frac{1}{x + iy} \]

So the \( \nu = \frac{1}{2} \) Pfaffian represents the LR part of a weak-coupling superconductor, taken right down to \( r = 0 \).
11.4.4 Topology in $k$-space:

Since $|u_k|^2 = 10^{-k}|z|^2 = 1$ and the overall phase is irrelevant, we can regard it as mapping $S$ and $u_k \to 1$ as $k \to \infty$, we can regard it as mapping $S^2 \to S^2$

$k$-space, with $\infty \to$ point

Gauss/Riemann sphere of complex number $e^{i \theta}$

The strong coupling phase has winding number $0$.

The weak coupling phase has winding number $1$.

(look near $k=0$ - single cover).