11.2.2 Ground state trial wavefunction for $\nu = \frac{1}{2}$

$$\Psi_{1/2}(z_1, \ldots, z_{2N}) = \underbrace{\mathrm{Pf}\,\frac{1}{z_i - z_j}}_{\substack{\text{Pairing in} \\ p_x + i p_y \text{ (i.e. } \ell = -1)}} \underbrace{\prod (z_i - z_j)^2 \, e^{-\frac{1}{4\ell_B^2}\sum |z_i|^2}}_{\text{Laughlin bosonic form}}$$

Note: Antisymmetric ✓

Non-singular

11.2.3 The filling fraction is in fact $\frac{1}{2}$, since the order of the polynomial is $2\left[\frac{N(N-1)}{2}\right]$ from the repulsive $(z_i - z_j)^2$ minus just $\frac{N}{2}$ from the Pfaffian.

Although this trial wave function is written for the lowest Landau level only, it can be used with minor modifications for $\nu = \frac{3}{2}, \frac{5}{2}, \ldots$, where Landau levels ~~be~~ well below the Fermi energy are completely full and essentially inert.

11.2.4 There are several diagnostics for the proposed universality class, given numerical simulation of electrons subject to a given Hamiltonian. They include:

a) Presence of a gap, complete spin polarization

b) Discrete ground state degeneracies depending on topology, e.g. for sphere, torus, + higher genus surfaces. This is similar in spirit to, but considerably more complicated than, the phenomena we saw in the Kitaev model. We'll analyze it later. Ref: cond-mat/0607743

c) N-dependent degeneracies for disc geometry (= edge states). We'll analyze it later. Ref: cond-mat/9906453

Topological field theories can be used to calculate numbers like in b) + c). These are theories with no conventional local degrees of freedom, but global structure. Example: Discrete gauge theories; pure Chern-Simons theories

↳ eq^{n} of motion: $f^{a}_{\mu\nu} = 0$!

11.2.5 There is good experimental evidence for a gapped* effectively spin-polarized state at $\nu = 5/2$. Numerical simulations with interactions thought to be realistic for the experiment in question (i.e. screening, Coulomb, ...) favor a Pfaffian ground state, according to the criteria just mentioned.

* Hall plateau

11.3 Quasiholes in the Pfaffian state

11.3.1 The quasiholes in this universality class are predicted to have remarkable properties, including both abelian and non-abelian quantum statistics. The abelian part is much as before ↳ i.e. for $\nu = 1/_{2m+1}$, with a twist we'll come to immediately; the non-abelian part is much as in $p_x + ip_y$ superconductors

11.3.2 According to the flux-trading heuristic, we can hope to relate $\nu = \frac{1}{2}$ to fermions with a gap in zero magnetic field. We're interested in effectively spinless fermions, so the simplest pairing possibility is p-wave, and to have a gap ↳ no nodes we like $p_x + ip_y$

11.3.3 In this procedure the electrons acquire fictitious flux $\Phi = \frac{2h}{q}$. For a paired state the elementary flux should be $\Phi = \frac{h}{2q}$. Thus we might expect to have quasiparticles that are "$\frac{1}{4}$ of an electron (or hole)", with charge $e^*/4$ and statistics $\pi/4$.

11.3.4 There is a subtlety in the statistics, since if 4 quasiparticles are interchanged with 4 others, we get $(e^{i\pi/4})^{16} = 1$ – bosons, not fermions!
[This works out for $1/m$ states: $(e^{i\pi/m})^{m^2} = e^{i\pi m} = -1$].
m odd!

{• Exactly this problem is repaired by the zero-mode.} (? conjecture)

• Four quasiholes + electron make a neutral fermion, which must also appear in the spectrum.

11.3.5 The wavefunction for quasiholes, that realizes all this intuition, is constructed as follows.

For a quasihole at z_0, modify the Pfaffian according to

$$\mathrm{Pf}\,\frac{1}{z_i - z_j} \longrightarrow \mathrm{Pf}\,\frac{(z_i - z_0) + (z_j - z_0)}{z_i - z_j}$$

This gives $\frac{1}{2}$ a Laughlin factor, and $\frac{1}{4}$ of an electron factor.
It boosts the orbital angular momentum of each pair by 1 unit.

11.3.6 For 2 quasiholes at z_1, z_2, the appropriate modification (which reduces to a Laughlin flux as $z_2 \to z_1$) is

$$\mathrm{Pf}\,\frac{1}{z_i - z_j} \to \mathrm{Pf}\,\frac{(z_i - z_1)(z_j - z_2) + (z_j - z_1)(z_i - z_2)}{z_i - z_j}$$

11.3.9 For four quasiparticles there appears, at first glance, to be three distinct possibilities for the numerator in the Pfaffian:

$$(z_i - \eta_1)(z_i - \eta_2)(z_j - \eta_3)(z_j - \eta_4) + (i \leftrightarrow j) \qquad \text{Write: } (12)(34)$$

$$(z_i - \eta_1)(z_i - \eta_3)(z_j - \eta_2)(z_j - \eta_4) + (i \leftrightarrow j) \qquad \equiv (13)(24)$$

$$(z_i - \eta_1)(z_i - \eta_4)(z_j - \eta_2)(z_j - \eta_3) + (i \leftrightarrow j) \qquad \equiv (14)(23)$$

But note $(12)(34) - (13)(24) = (z_i - z_j)^2 (\eta_1 - \eta_4)(\eta_2 - \eta_3)$

vanishes as $z_i \to z_j$, symmetric, order 2

vanishes as $\eta_1 \to \eta_4$ or $\eta_2 \to \eta_3$

so

$$\frac{(12)(34) - (13)(24)}{(12)(34) - (14)(23)} = \frac{(\eta_1 - \eta_4)(\eta_2 - \eta_3)}{(\eta_1 - \eta_3)(\eta_2 - \eta_4)} \qquad (\text{"cross-ratio"})$$

is independent of z!

Thus there are only 2 independent configurations. – for two electrons!

(note: Pfaffian is non-linear)

~~Other argument is~~

For larger number of electrons, with a little more work we prove that

$$\mathrm{Pf}_{(12)(34)} - \mathrm{Pf}_{(14)(23)} = \frac{\eta_{14}\eta_{23}}{\eta_{13}\eta_{24}} \left(\mathrm{Pf}_{(12)(34)} - \mathrm{Pf}_{(13)(24)} \right)$$

i.e. $\dfrac{(\eta_1-\eta_4)(\eta_2-\eta_3)}{(\eta_1-\eta_3)(\eta_2-\eta_4)}$

in general, (C. Nayak + F.W., NP B479, 529)

11.3.10 The generalization to more quasiholes gives a basis as follows:

[illegible]

$$\mathrm{Pf}_{(\eta_a \cdots \eta_x)(\eta_\alpha \cdots \eta_\chi)}$$

subject to the restriction that η_1, η_2 appear in one Latin and one Greek grouping, as do η_3, η_4; η_5, η_6; etc.

This gives $2^n/2 = 2^{n-1}$ basis states, for $2n$ quasiholes

1st member ~~in or out~~ Greek or Latin

overall interchange Greek ↔ Latin

Interpretation: η_1, η_2 are "paired", in the sense that as $\eta_1 \to \eta_2$ we get a conventional (Laughlin) vortex.

~~[illegible]~~ A distant vortex does not resolve Cooper pairs. As we bring $\frac{1}{2}$-vortices in, ~~two~~ at a time, each electron in a pair must see one of them. ~~[illegible]~~ The first choice is abitrary, but subsequent ones build unambiguously on the previous

$$\begin{array}{ccccc} 1 & 2 & & & \\ \downarrow & & & & \\ & & & \underline{135} & \underline{246} \\ 13 & 24 & \longrightarrow \text{or} & 136 & 254 \\ \text{or} & & & & \\ & & & \underline{145} & \underline{236} \\ 14 & 23 & \longrightarrow \text{or} & 146 & 235 \end{array}$$

etc.

11.3.11 In principle one can now use the Berry ~~p~~hase to compute braiding, including the nonabelian factor. But this has never been done directly!! [Project]

(I think it's very doable, though.)

Using conformal field theory, we showed the spinor representation emerged, as for Majorana modes. [original insight: Moore + Read]

~~Now We'll now relate~~ it

It's instructive to approach the situation in different ways.

11.4 Relation to $p_x + ip_y$ superconductor

(Read + Green
cond-mat. 19906453)

11.4.1 Let's recall the relevant essence of BCS theory.

$$K_{eff.} = \sum_{\vec{k}} \varepsilon_{\vec{k}} c^{+}_{\vec{k}} c_{\vec{k}} + \frac{1}{2}\left(\Delta^{*}_{k} c_{-k} c_{k} + \Delta_{k} c^{+}_{k} c^{+}_{-k}\right)$$

$$\varepsilon_k = \epsilon_k - \mu = \frac{k^2}{2m} - \mu$$

$$\Delta_{\vec{k}} \sim \hat{\Delta}(k_x - ik_y) \text{ for small } k$$
$$\to 0 \text{ for } k \to \infty$$

$$|\Omega\rangle = \prod_k \left(u_k + v_k c^{+}_{k} c^{+}_{-k}\right)|0\rangle$$

$$\alpha_k = u_k c_k - v_k c^{+}_{-k} \; ; \; \{\alpha_k, \alpha^{+}_{k'}\} = \delta_{kk'} \; ; \; \alpha_k |\Omega\rangle = 0$$
$$\alpha^{+}_{k} = u^{*}_{k} c^{+}_{k} - v^{*}_{k} c_{-k}$$

Demanding $[\alpha_k, K_{eff.}] = E_k \alpha_k$, we diagonalize $K_{eff.} = \sum_k E_k \alpha^{+}_{k} \alpha_k$

$E_k \geq 0$

The wave equation (Bogoliubov - de Gennes) is

$$E_k u_k = \varepsilon_k u_k - \Delta_k^* v_k$$

$$E_k v_k = -\varepsilon_k v_k - \Delta_k u_k$$

which leads to

$$E_k = \sqrt{\varepsilon_k^2 + |\Delta_k|^2}$$

$$v_k / u_k = -(E_k - \varepsilon_k)/\Delta_k^*$$

$$|u_k|^2 = \frac{1}{2}\left(1 + \frac{\varepsilon_k}{E_k}\right) = 1 - |v_k|^2$$

11.4.2 The ground state can be written (up to a phase)

$$|\Omega\rangle = \prod_k (u_k + v_k c_k^+ c_{-k}^+)|0\rangle = \prod_k |u_k|^{1/2} \exp\left(\frac{1}{2}\sum_k g_k c_k^+ c_{-k}^+\right)|0\rangle$$

with $g_k \equiv v_k / u_k$ . .

Expanding and passing to real space, we find in the N ~~even~~ ~~electron~~ sector

$$\Psi(r_1, \dots r_N) \propto \underbrace{\sum \operatorname{sgn}\pi \prod_{i=1}^{N/2} \tilde{g}(r_{\pi(2i-1)} - r_{\pi(2i)})}_{\text{Pfaffian!}}$$

[note: N is even!]

11.4.3 Distinguish:

strong pairing ("molecular BEC"): $\varepsilon_k > 0$ as $k \to 0$

$\Rightarrow |u_k| \to 1, |v_k| \to 0$ unoccupied

weak pairing (BCS - but not necessarily very weak coupling): $\varepsilon_k < 0$ as $k \to 0$

$\to |u_k| \to 0, |v_k| \to 1$

In strong pairing, $g_k = v_k / u_k \to \propto k_x - ik_y$ as $k \to 0$. It is well-behaved at ∞, and analytic, so $g(r) \sim e^{-r/r_0}$.

In weak pairing, $g_k \to \propto \dfrac{1}{k_x - ik_y}$ as $k \to 0$.

$$g(r) \propto \int \frac{d^2k}{k_x - ik_y} e^{ikr} \xrightarrow[r\to\infty]{} \frac{1}{x+iy} \; !$$

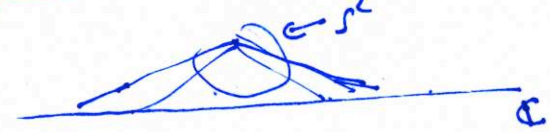
So the $\nu = 1/2$ Pfaffian represents the LR part of a weak-~~coupling~~ pairing k_x+ik_y superconductor, taken right down to $r=0$.

11.4.4 Topology in k-space:

~~For~~ Since $|u_k|^2 = |v_k|^2 = 1$ and the overall phase is irrelevant, we can regard it as mapping, ~~S~~ and $u_k \to 1$ as $k \to \infty$, we can regard it as mapping $S^2 \xrightarrow{u_k/v_k} S^2$

k-space, with $\infty \to$ point

Gauss/Riemann sphere of complex numbers



The strong ~~coupling~~ pairing phase has winding number 0.

or degree

The weak coupling phase has winding number 1.

(look near $k=0$ – single cover).