Lecture 12: Majorana Zero Modes, Edge States, Vortices, Topology (Loose ends + Baby Skyrmions)

12.1 $p_x + ip_y$ superconductor

12.1.1 Last time we finished up showing that the Pfaffian factor which appears in the $\nu = 1/2$ (i.e., 5/2) quantum Hall state can be interpreted as the wave function for electrons in a $p_x + ip_y$ weak pairing superconductor.

So it behooves us to study the mode structure for fermions in this case, especially the issues of edge states and 0-modes on vortices.

The mathematical structure of this problem is essentially isomorphic* to that of the non-abelian Kitaev model phase, and to an interesting Skyrmion problem.

* Actually, one extra insight is required.

12.1.2 Recall:

Definitions (BCS)

$$\mathcal{E}_{\vec{k}} = \epsilon_{\vec{k}} - \mu = \frac{k^2}{2m} - \mu$$

$$\alpha_{\vec{k}} = u_{\vec{k}} c_{\vec{k}} - v_{\vec{k}} c^{\dagger}_{-\vec{k}}$$ [remark on pairing instability from degenerate perturbation theory]

Bogoliubov-de Gennes equations

$$\boxed{\begin{aligned} E_{\vec{k}} u_{\vec{k}} &= \mathcal{E}_{\vec{k}} u_{\vec{k}} - \Delta^*_{\vec{k}} v_{\vec{k}} \\ E_{\vec{k}} \vec{v}_k &= -\mathcal{E}_{\vec{k}} v_{\vec{k}} - \Delta_{\vec{k}} u_{\vec{k}} \end{aligned}}$$

They arise by demanding

$$[\alpha_{\vec{k}}, H_{eff.}] = E_{\vec{k}} \alpha_{\vec{k}}$$

with the pairing Hamiltonian

$$H_{eff.} = \sum_{\vec{k}} \mathcal{E}_{\vec{k}} c^{\dagger}_{\vec{k}} c_{\vec{k}} - \Delta_k c_{-\vec{k}} c_{\vec{k}} - \Delta^*_k c^{\dagger}_{\vec{k}} c^{\dagger}_{-k}$$

Solution

$$E_{\vec{k}} = \sqrt{\mathcal{E}^2_{\vec{k}} + |\Delta_k|^2}$$

$$v_{\vec{k}} / \vec{u}_k = -(E_k - \epsilon_k) / \Delta^*_k \equiv g_{\vec{k}}$$

$$|u_k|^2 = \frac{1}{2}\left(1 + \frac{\epsilon_k}{E_k}\right) = 1 - |v_k|^2$$

12.1.3 Straight Edge

We take

$$\Delta = \hat{\Delta}(k_x - ik_y)$$

as an external field, with $\hat{\Delta}$ constant. (Properly, $\hat{\Delta}$ should be determined self-consistently, by solving the gap equation. But we are O.K. for our topological purposes as long as it's smooth.]

The B-dG equations become

$$E_k \psi_k = (\sigma_3 \varepsilon_k - \sigma_1 \hat{\Delta} k_x - \sigma_2 \hat{\Delta} k_y)\psi_k, \quad \text{with } \psi_k \equiv \begin{pmatrix} u_k \\ v_k \end{pmatrix}$$

if we assume $|k|$ small, writing $\varepsilon_k \simeq -\mu$, and go to position space, we get the Dirac equation

$$i\frac{\partial}{\partial t}\psi = \left(-\mu\sigma_3 + i\sigma_1 \hat{\Delta}\frac{\partial}{\partial x} + i\sigma_2 \hat{\Delta}\frac{\partial}{\partial y}\right)\psi$$

identifying: $\gamma_0 = +\sigma_3$, $\gamma_1 = i\sigma_2$, $\gamma_2 = -i\sigma_1$, $\mu = -m$

It is helpful to use instead the γ-matrices

$$\gamma_0 = +\sigma_2 \,,\ \gamma_1 = i\sigma_3 \,,\ \gamma_2 = -i\sigma_1$$

Since these are all imaginary, the Dirac equation becomes entirely real. This is called a Majorana basis. We can transform to it by defining $\tilde{\Psi} = U\Psi$ with $U = \frac{1 i\sigma_1}{\sqrt{2}}$ $[U^{-1}\sigma_2 U = -\sigma_3 \,,\ U^{-1}\sigma_3 U = +\sigma_2 \,,\ U^{-1}\sigma_1 U = \sigma_1]$

Now we consider an edge along the y-axis:

An edge forces electrons ~~...~~ not to cross. It can be enforced by $\mu \to \infty$.

First taking $E = k_y = 0$, we get simply (taking $\hat{\Delta}$ real & > 0)

$$\hat{\Delta}\, i \frac{\partial v}{\partial x} = \mu(x)\, u$$

$$\hat{\Delta}\, i \frac{\partial u}{\partial x} = -\mu(x)\, v$$

solve with $v = iu = u^*$ (in Ψ formalism)

$$-\hat{\Delta} \frac{\partial u}{\partial x} = \mu(x)\, u$$

$$u(x) \propto \exp \int_0^x -\frac{\mu(s)}{\hat{\Delta}}\, ds \qquad \text{converges - as in JR domain wall}$$

In $\tilde{\Psi}$ form

$$-\mu \sigma_2 \tilde{\Psi} + i \sigma_1 \hat{\Delta} \frac{\partial}{\partial x} \tilde{\Psi} = 0 \quad \text{decouples}$$

$$-\mu \tilde{\Psi}_2 + \hat{\Delta} \frac{\partial \tilde{\Psi}_2}{\partial x} = 0 \quad \Rightarrow \text{non-normalizable}$$

$$\mu \tilde{\Psi}_1 + \hat{\Delta} \frac{\partial \tilde{\Psi}_1}{\partial x} = 0 \quad \tilde{\Psi}_1 \propto \exp \int_0^x -\frac{\mu(s)}{\hat{\Delta}}\, ds$$

Now for finite k_y

$$(E_k + \sigma_3 \hat{\Delta} k_y)\tilde{\Psi} = (-\sigma_2 \xi_k - \sigma_1 \hat{\Delta} k_x)\tilde{\Psi}$$

so to "continue" the $E_k = 0$ ($\tilde{\Psi}_2 = 0$) mode we just take $k_y = -E_k$. The modes ~~go~~ propagate only one way! $\Rightarrow$ Chiral Majorana edge states.

12.1.4 Vortex

First, we eliminate the gauge field by means of a "singular" gauge transformation. For wave functions that vanish at the core, the effect is to enforce antiperiodic boundary conditions ($L = \frac{1}{2}$ odd integer). [This of course is for charge e, flux $\frac{h}{2e}$.

$$A_\phi = \frac{h}{2e} \frac{1}{2\pi} \Rightarrow \tilde{A}_\phi = A_\phi - \partial_\phi \Lambda = 0 \quad \text{with } \Lambda = \phi \frac{\hbar}{2e}$$

$$\tilde{\Psi} = e^{i\Lambda e/\hbar} \Psi \Rightarrow \frac{\tilde{\Psi}(2\pi)}{\tilde{\Psi}(0)} = -\frac{\Psi(2\pi)}{\Psi(0)} = -1$$

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To enforce vanishing, $\mu \to \infty$ at core center $r = 0$.

In original formalism, BdG with $\Delta = \hat{\Delta}\,\cancel{e^{i\theta}}$, $E = 0$

after gauge transformation

$$\hat{\Delta}\, i e^{i\theta} \left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right) v = \mu u$$

$$\hat{\Delta}\, i e^{-i\theta} \left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \theta}\right) u = -\mu v$$

If we want consistent angular dependence, must take $u \propto e^{i\theta/2}$, $v \propto e^{-i\theta/2}$

With standard 2d form $\sim \frac{1}{\sqrt{r}}$, write

$$u = e^{-i\pi/4} \frac{f(r)}{\sqrt{r}} e^{i\frac{\theta}{2}}$$

$$v = -e^{i\pi/4} \frac{f(r)}{\sqrt{r}} e^{-\frac{i\theta}{2}} = i u^*$$

Then we find for f

$$\hat{\Delta} \frac{\partial f}{\partial r} = \mu f \qquad f(r) \propto \exp \int_{r_0}^{r} -\frac{\mu(s)}{\hat{\Delta}}\, ds$$

12.1.5 Disc

For a large disc we put together ~~edge~~ straight-edge solutions to find modes with $E \sim m/R$, $m = \frac{1}{2}$ odd integer in absence of vortex

For vortex, $E \sim m/R$ including $m=0$ true 0-mode, as we've seen.

Puzzle: Where does this mode reside - at core or on edge?

N.B. The ~~[illegible]~~ gauge-transformed vortex equation correspond to $\hat{\Delta} \propto k_x - ik_y$, with no additional phase.

$$\frac{\partial}{\partial x} - i\frac{\partial}{\partial y} = e^{-i\theta}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \theta}\right), \text{ etc.}$$

das Sarma + co. effectively considered the $e \to 0$ case, and did not make a gauge transformation. Therefore their 0-mode appeared in $m=1$ rather than $m=\frac{1}{2}$!

12.2 Topological Dictionary (Pidgin version)

12.2.1 k-space topology – sphere version

The structure of the BdG / Dirac equation

$$E_k \tilde{\Psi} = (\varepsilon_k \sigma_2 - \sigma_1 \mathrm{Re}\,\Delta_k - \sigma_3 \mathrm{Im}\,\Delta_k)\tilde{\Psi} \qquad \left(E_k = \sqrt{\varepsilon^2 + |\Delta|^2} \right)$$

encodes a mapping $\vec{k} \to S^2$ by

$$\vec{k} \to \frac{1}{E_k}(\underset{\substack{\uparrow \\ \text{coff. of} \\ \sigma_2}}{\varepsilon_k}, \underset{\substack{\uparrow \\ \text{coff. of} \\ \sigma_1}}{\mathrm{Re}\,\Delta_k}, \underset{\substack{\uparrow \\ \text{coff. of} \\ \sigma_3}}{\mathrm{Im}\,\Delta_k})$$

The weak pairing / strong pairing vortex sol'ns. correspond to this map being a single covering $S^2 \to S^2$! (with trivial structure at $k \to \infty$). In general, the topology of maps is classified by degree (# of coverings, with sign)

12.2.2 k-space topology – complex number / gauge field version

We can organize the information in a unit vector into a complex number (+ vice versa) by
(possibly ∞)

$$(n_1, n_2, n_3) \longleftrightarrow \frac{n_2 + i n_3}{n_1} \equiv \rho$$

The degree is the same.

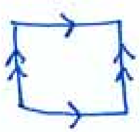
From ρ we can construct the gauge field $\frac{1}{i\rho} \nabla_k \rho$.
This is ill-defined at $\rho = 0$ or ∞, however. These points may enclose flux! The total flux (= monopole #) is closely related to the degree. [exercise]

12.2.3 The degree can be calculated from the integrated Jacobian ~~Index~~

$$\int \epsilon^{abc} \epsilon^{\beta\gamma} n^a \frac{\partial n^b}{\partial k^\beta} \frac{\partial n^c}{\partial k^\gamma} d^2 k$$

[exercise, + fix normalization]

12.2.4 The sphere can be considered as a special case of the torus:



torus: identify opposite sides
sphere: identify everything

The ~~twist~~ degree for a sphere goes into the Chern number for a torus [exercise]
[actually, we did it: Chern # = total flux]

We've demonstrated the connection of 0-modes with Chern # in special cases. The relationship is general (index theorem).

Physicist's proof: Laughlin argument!

Project/term paper: index theorems in CM physics.