Lecture 15: Quantum Spin Hall Effects

15.0 technological motivations:

"spintronics" for cool calculation

problem of injection

theoretical motivations:

brings in more topology

dissipationless $\Rightarrow$ reversible (even quantum) computing

$\Leftarrow$, more accurately!

15.1 Kramers' theorem

This classic consequence of T-symmetry will play an important role, so I'll review it.

15.1.1 Since Schrödinger's equation takes the form

$$i \frac{\partial \psi}{\partial t} = H\psi$$

If we want T-reversal symmetry of H to correspond to symmetry of the equation under $t \to -t$, we must reverse the sign of i. Thus T is implemented by an antiunitary transformation

$$\psi \longrightarrow K_0 \psi^*$$

$$\alpha\psi \rightarrow \alpha^* K_0 \psi^*$$

15.1.2 For spin-$\frac{1}{2}$ we take $K_0 = i\sigma_y$, in order that the expectation of the spin reverses sign

$$T \langle \psi | \vec{\sigma} | \psi \rangle \rightarrow \langle \psi | K_0^\dagger \vec{\sigma}^* K_0 | \psi \rangle \underset{K_0 = i\sigma_y}{=} \langle \psi | -\vec{\sigma} | \psi \rangle$$

15.1.3 The underlying mathematical reason is that the spin-$\frac{1}{2}$ representation is pseudo-real: that is, it is equivalent to its complex conjugate, but by a non-trivial operation, so it cannot be taken real. The required transformation is indeed with $\varepsilon^{\alpha\beta} = (i\sigma_y)^{\alpha\beta}$

[φ^{α}] transforms with $\begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$

$\varphi^{\alpha *}$ transforms with $\begin{pmatrix} \alpha^* & \beta^* \\ -\beta & \alpha \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

i.e. $\epsilon^{\alpha\beta}\varphi^{*\beta}$ transforms as φ^{α},

Aside: this trick is important in the minimal standard model, so that one ~~do~~ Higgs doublet ~~can~~ (and its complex conjugate) can give mass to fermions with different hypercharge.

15.1.4 It follows that $K_0^2 = -1$ (!)

The same holds for any half-odd-integer spin

[Pf: Represent it as a symmetric spinor with and odd # of indices; we get an odd number of copies of the spin-$\frac{1}{2}$ K_0]

15.1.5 Thus for $\frac{1}{2}$-odd integer spin T invariant systems (including interacting systems with odd # of spins $\frac{1}{2}$!) φ and $K\varphi$ are degenerate, orthogonal states:

$$\text{In general} \quad \langle K\varphi, K\psi\rangle \underset{\text{antiunitarity}}{=} \langle \varphi, \psi\rangle^* = \langle \psi, \varphi\rangle ;$$

$$\langle K^2\varphi, K\varphi\rangle \underset{\substack{\text{antiunitarity} \\ \text{- pull away } K}}{=} \langle \varphi, K\varphi\rangle$$

$$\shortparallel$$

$$\langle -\varphi, K\varphi\rangle$$

$$\text{so} \quad \langle \varphi, K\varphi\rangle = 0.$$

This is Kramers theorem. $\varphi, K\varphi$ are called a Kramers doublet.

15.2 Quantum spin Hall effect in graphene

15.2.1 Organize modes around two critical k-points. Around each we have a 2-component spinor, say concentrated on A, B lattice sites. Adding an index for one site vs. the other, we get a 4-component spinor.

$\vec{\tau}$ acts in $k^*/-k^*$ space, $\vec{\sigma}$ acts in A/B space.

Dirac equation(s) takes the form

$$H_0 = -i v_F \psi^\dagger (\sigma_x \tau_z \partial_x + \sigma_y \partial_y) \psi$$

N.B.: $\tau_z = \pm 1$ correspond to different orientations $\hat{k}_x \times \hat{k}_y$.

Terms proportional to σ_z or $\sigma_z \tau_z$ would open a gap at $\vec{q} = 0$, but they are forbidden by inversion I and T respectively.

I $\quad I_x : \sigma_y \quad (+\, x \to -x)$

$\quad I_y : \sigma_x \quad (+\, y \to -y)$

$\quad T : \tau_x \quad (+\, i \to -i)$, 15.2.2 Introduce spin $\vec{s}$.

$$H_{spin\text{-}orbit} = \Delta_{SO} \psi^\dagger \sigma_z \tau_z s_z \psi$$

obey all the symmetries, and opens gaps. It also maintains mirror symmetry of reflection in plane $\begin{pmatrix} s_z \to s_z \\ s_x, s_y \to -s_x, -s_y \end{pmatrix}$. Relaxing that constraint allows the "Rashba" term

$$H_R = \lambda_R \left(\psi^\dagger (\sigma_x \tau_z s_y - \sigma_y s_x) \psi \right) \qquad (\text{``}\equiv\text{''} \ (\vec{S}\times\vec{p})\cdot\hat{z}E)$$

$\vec{p} \to \vec{k}$ (rotated)

This could be induced by a normal electric field

15.2.3 For $\lambda_R = 0$:

$\tau_z = 1 \quad s_z = 1$

$$H = \begin{pmatrix} \Delta & v_F(p_x - ip_y) \\ v_F(p_x + ip_y) & -\Delta \end{pmatrix}$$

$\tau_z = 1 \quad s_z = -1$

$$\begin{pmatrix} -\Delta & v_F(p_x - ip_y) \\ v_F(p_x + ip_y) & \Delta \end{pmatrix}$$

$m \leftrightarrow -m$

orientation change

$\tau_z = -1 \quad s_z = 1$

$$\begin{pmatrix} -\Delta & v_F(-p_x - ip_y) \\ v_F(-p_x + ip_y) & \Delta \end{pmatrix}$$

$\tau_z = -1 \quad s_z = -1$

$$\begin{pmatrix} \Delta & v_F(-p_x - ip_y) \\ v_F(-p_x - ip_y) & -\Delta \end{pmatrix}$$

In all cases eigenvalues $\pm E(p)$, $E^2 = v_F^2 \vec{p}^{\,2} + \Delta^2$

The same structures arose in Haldane's model.

15.2.4 Since there is no T-violation, there can be no net 0-field Hall effect. (That is, no dissipationless effect from filled bands below gap!) There are $\sigma_{xy} = \pm \frac{e^2}{h}$ contributions from the two critical momenta.

$$E \xrightarrow{\;j\uparrow\;} \qquad \xrightarrow[T]{} \qquad \xrightarrow[E]{\;j\downarrow\;}$$

15.2.5 However T also changes $S_z \to -S_z$, so a spin current

$$J_s = \frac{\hbar}{2e}\left(J_\uparrow - J_\downarrow \right)$$

is allowed by symmetry (also P)

$$E \xrightarrow{\;j^\uparrow \uparrow \;\Rightarrow\; J_s \uparrow\;} \qquad \xrightarrow[\text{invert in } y \text{ or: } T]{} \qquad E \xrightarrow{\;j^\downarrow \downarrow \;\Rightarrow\; J_s \uparrow\;}$$

In fact this occurs, since the different critical momenta-points are related by: change the orientation and the spin direction.

15.2.6 The upshot is a quantized spin Hall conductivity

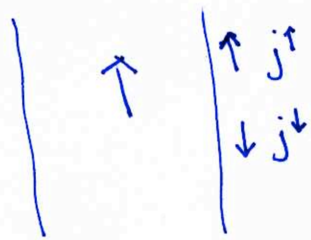
↑ but see below!

$$\sigma^s_{xy} = \frac{e}{2\pi}$$

The quantization is not accurate, however, since spin is not strictly conserved!

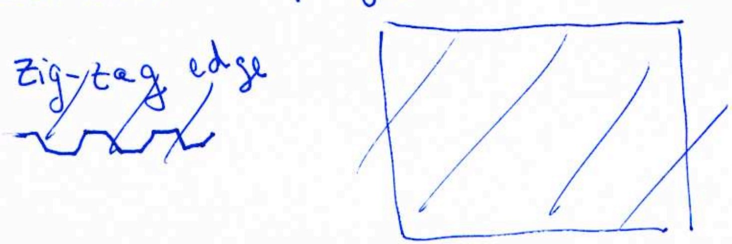
15.2.7 For $\lambda_R = 0$ (conserved spin) the existence of gapless edge states follows from Laughlin's argument. They are chiral and spin-filtered.

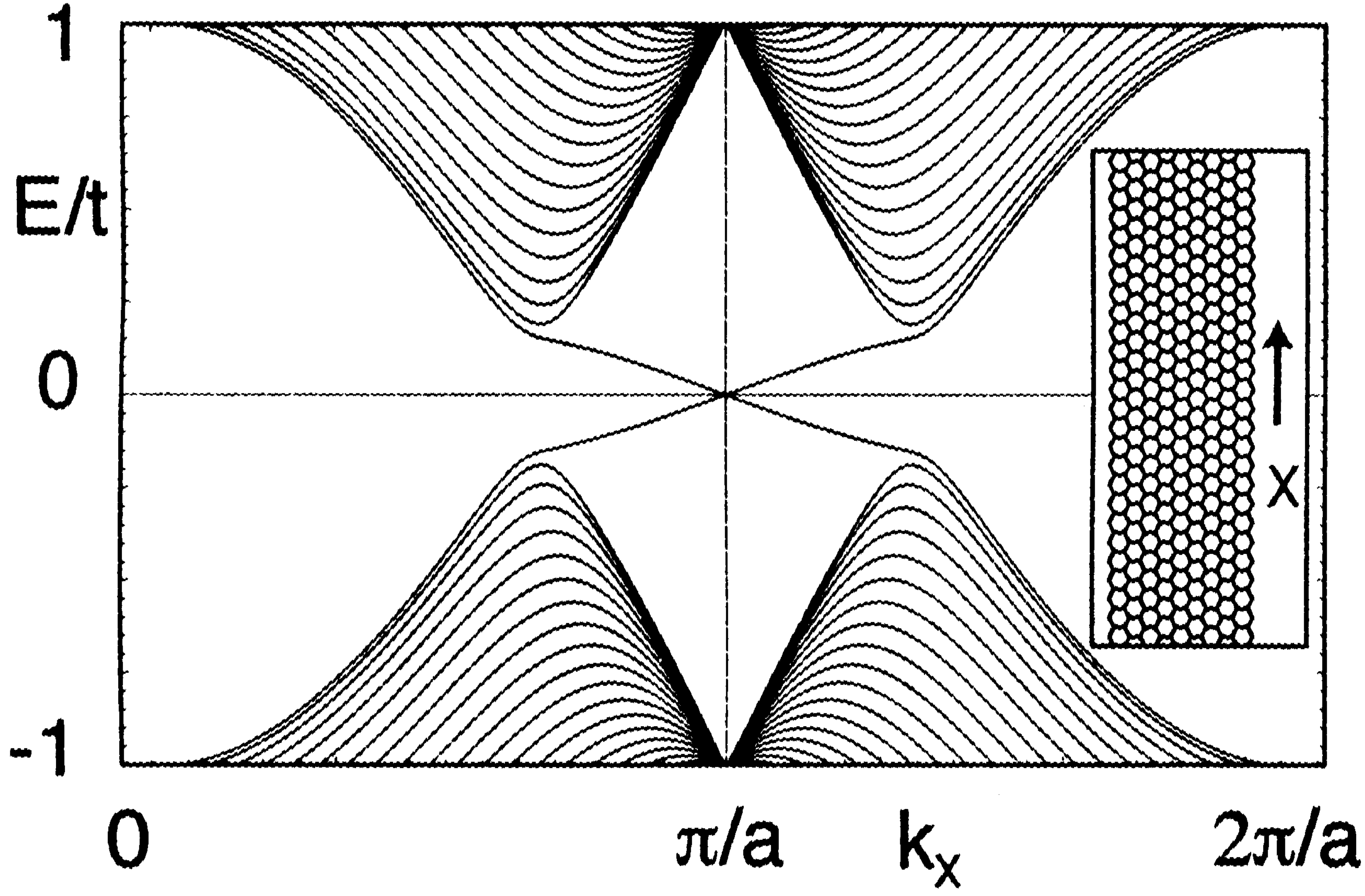
↳ [Put quantum of flux through hole in annulus. Faraday ⇒ electric field ⇒ Hall charge transfer. Bulk grd. state unchanged ⇒ gapless donors & acceptors at edge.]



15.2.8 To see the edge states explicitly, for a ribbon geometry, one must have a model for the entire Brillouin zone (not just the critical momentum cones). Haldane's term (with $\phi = 0$) + nearest-neighbor hopping is enough.

Results (without proof)




1
E/t
0
-1
0
π/a
k_x
2π/a
x

15.2.9 Turning on a ~~gap~~ Rashba term will not alter this picture qualitatively, as long as the gap remains open.

Indeed, the pair at π/a is a Kramers doublet.

15.2.10 T-invariant scattering (e.g., from impurities) cannot reverse the currents, due to Kramers again ($k\uparrow, -k\downarrow$ Kramers doublet $\xRightarrow[\text{proof}]{\text{similar}}$ $\langle \phi | H | K\phi \rangle = 0$)

T invariant

This applies to 1-body operators but not to interactions, which can reverse the currents. These however are irrelevant in the RG sense.

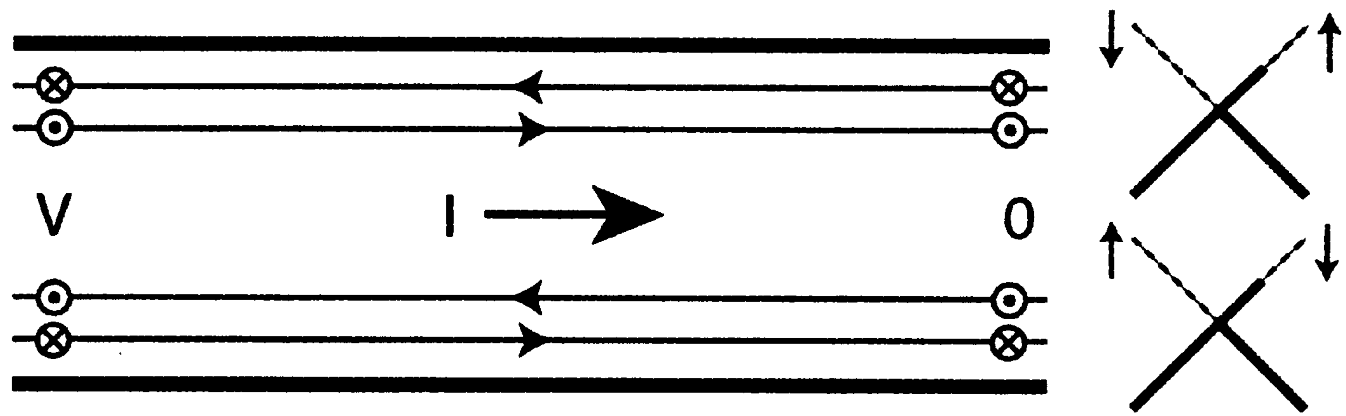
15.2.11 Numerically for graphene

$\lambda_R \sim .5$ mK

$\Delta_{SO} \sim 2.4$ K $\xrightarrow[\text{renormalization}]{}$ 15 K

reference: C. Kane + E. Mele

cond/mat/0411737



(a)