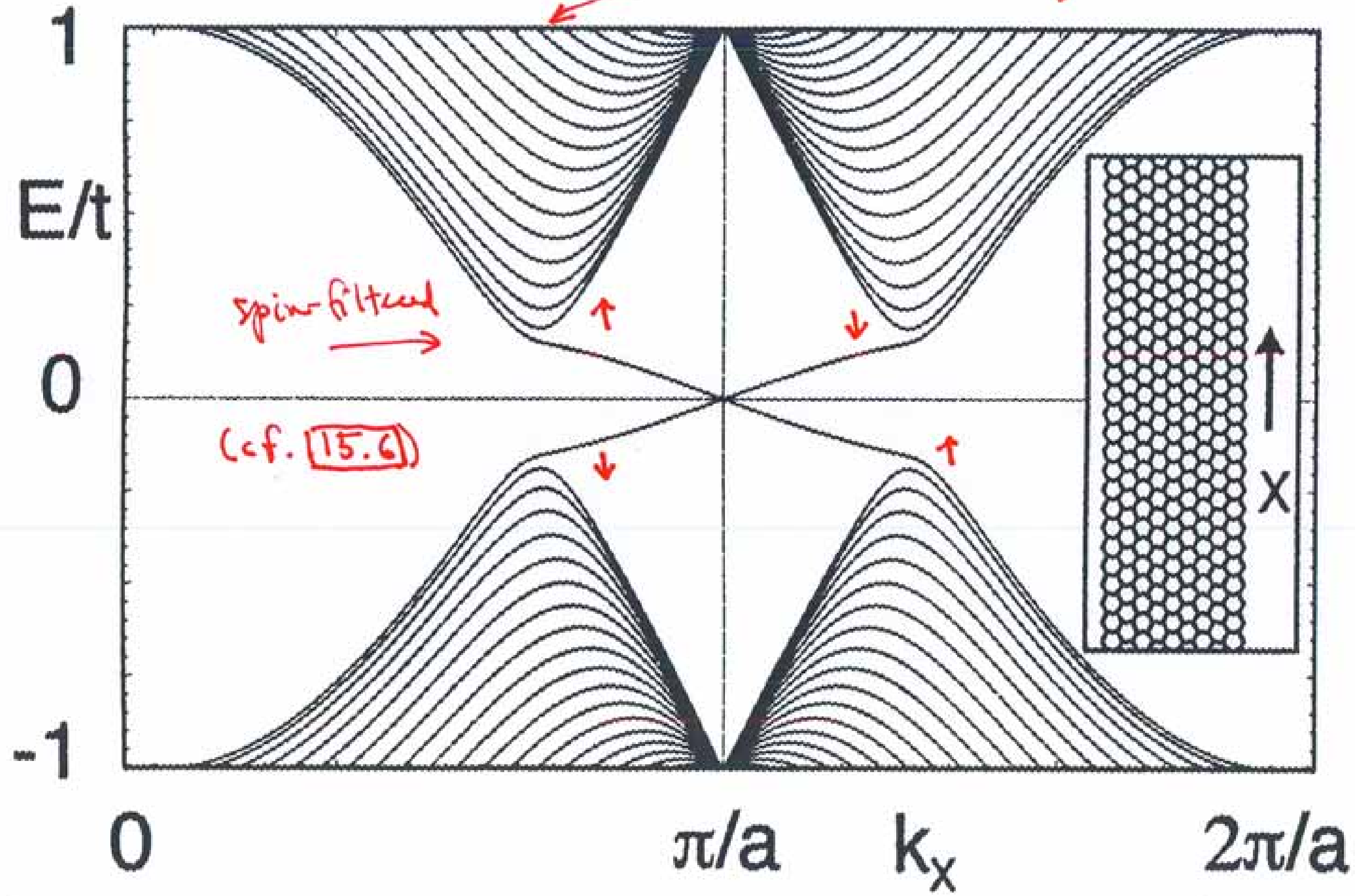
Scholium on the figures . . .

(Dirac: "I understand ~~and~~ an equation when I can anticipate the properties of its solution without actually solving it.")

$\boxed{15.13}$

projections of K, K'
"cones"

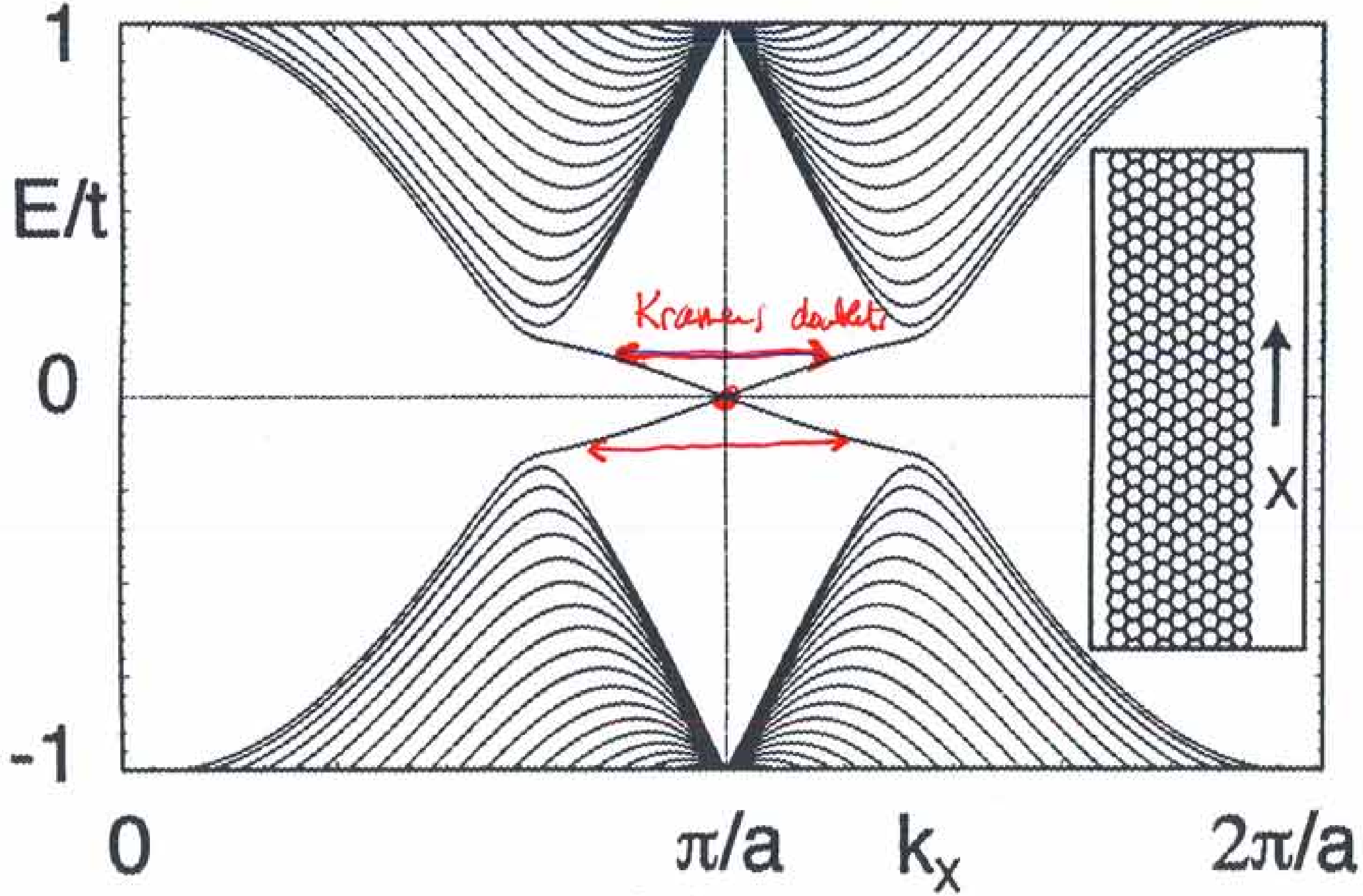


spin-filtered

(cf. $\boxed{15.6}$)

note: i) K_y discretized

ii) gap opened by Δ_{SO}

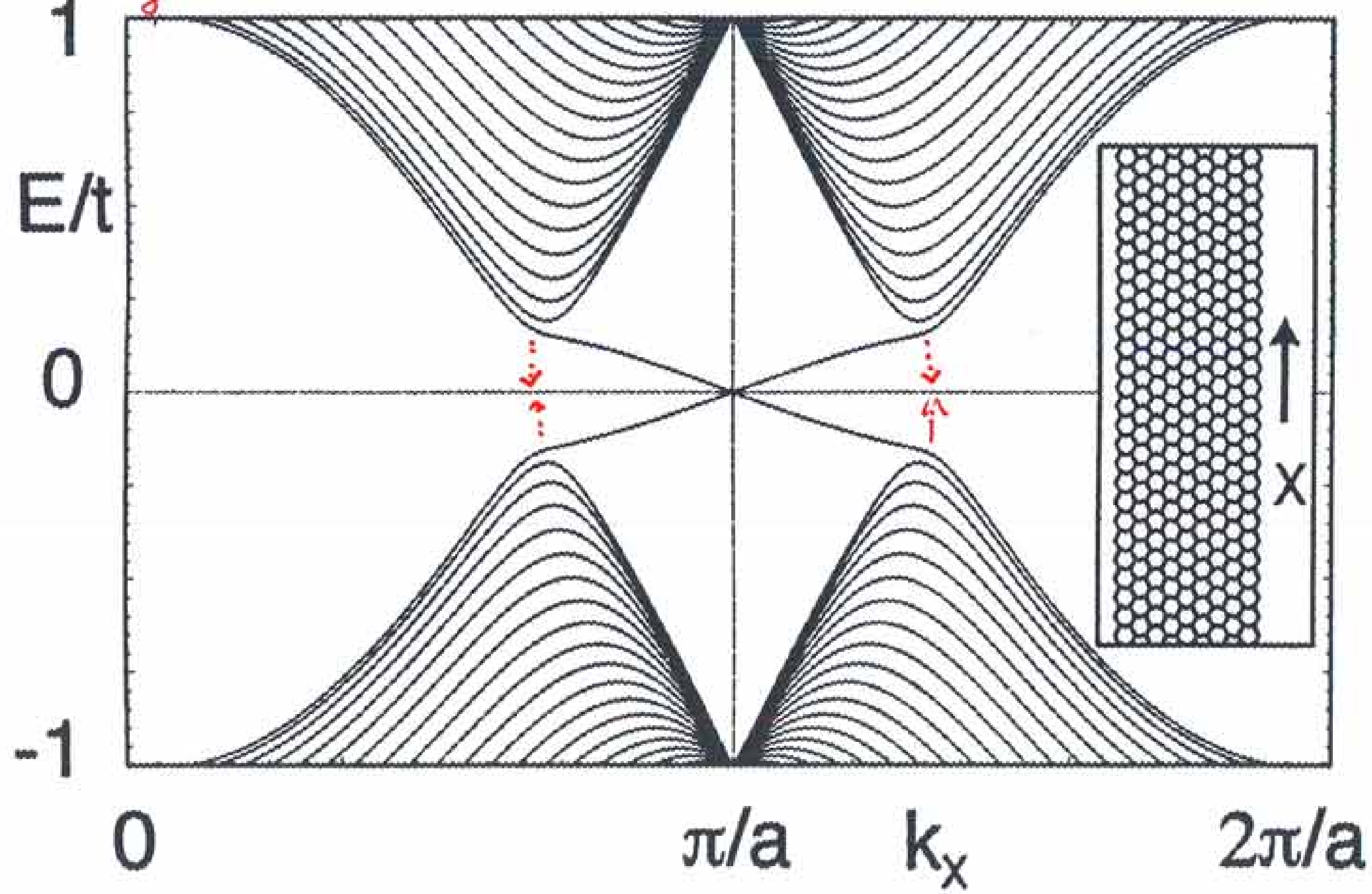


- At π/a ("=" $-\pi/a$) the doublet is a crossing.
- The crossing cannot be lifted by T-invariant perturbations

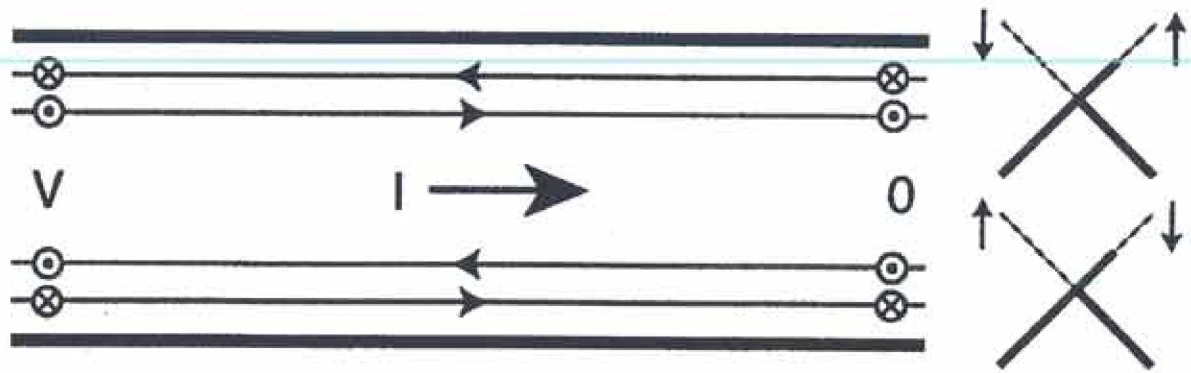
15.15

As $\Delta_{SO} \to 0$ the cones come down and the 0-modes become exactly flat.

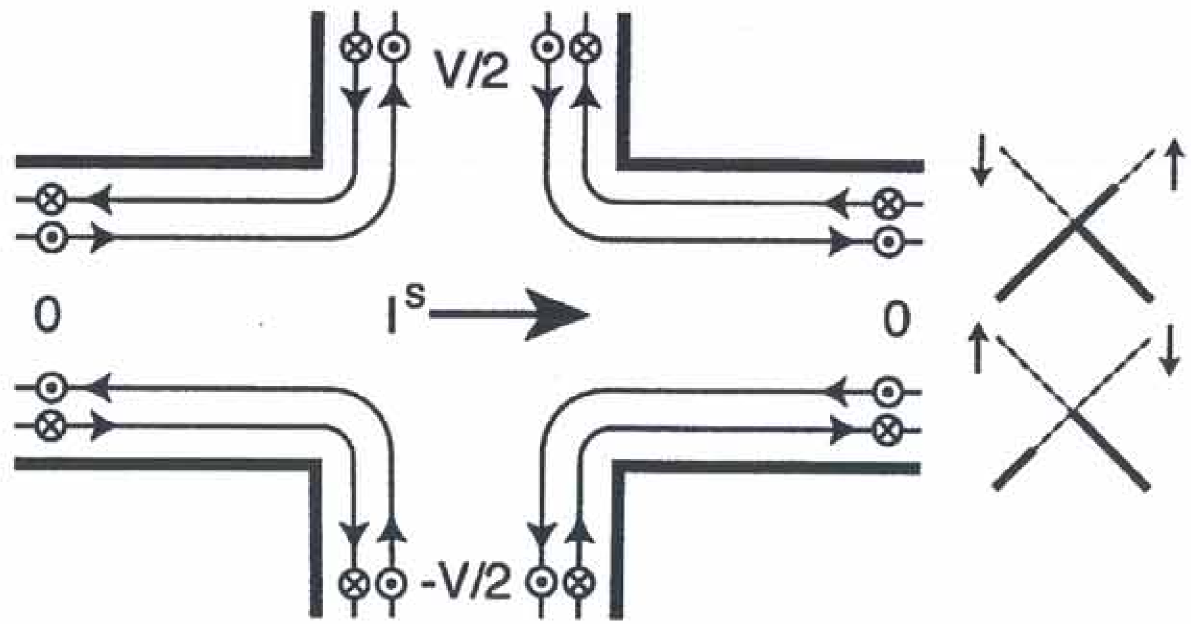
This enhanced density of states has been observed by STM.



The spin-up (⊙) current wants the wall to its left.
The spin-down (⊗) current wants the wall to its right



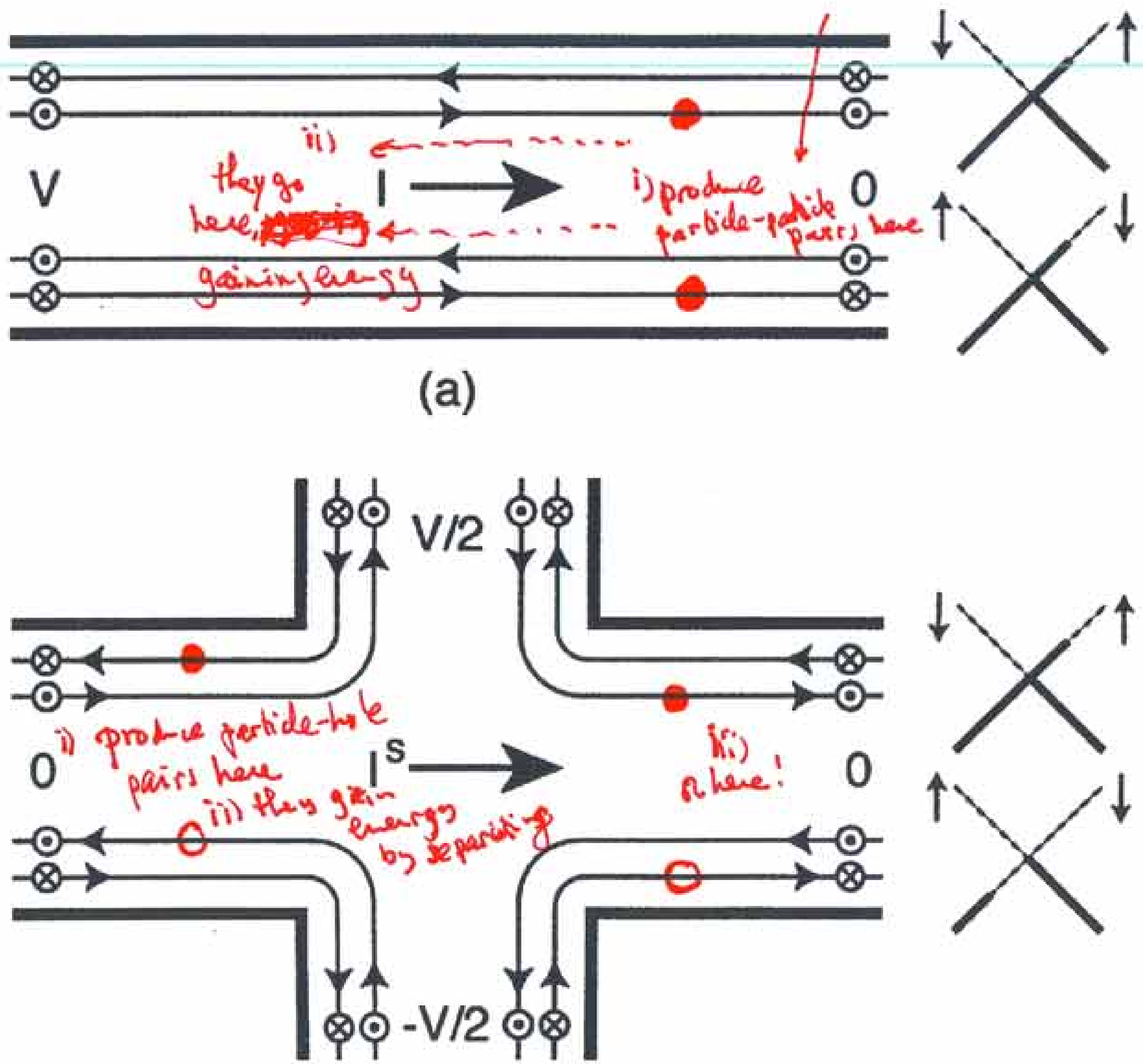
(a)



The "sign" of this effect depends on the sign of Δ_{so} (a material property).

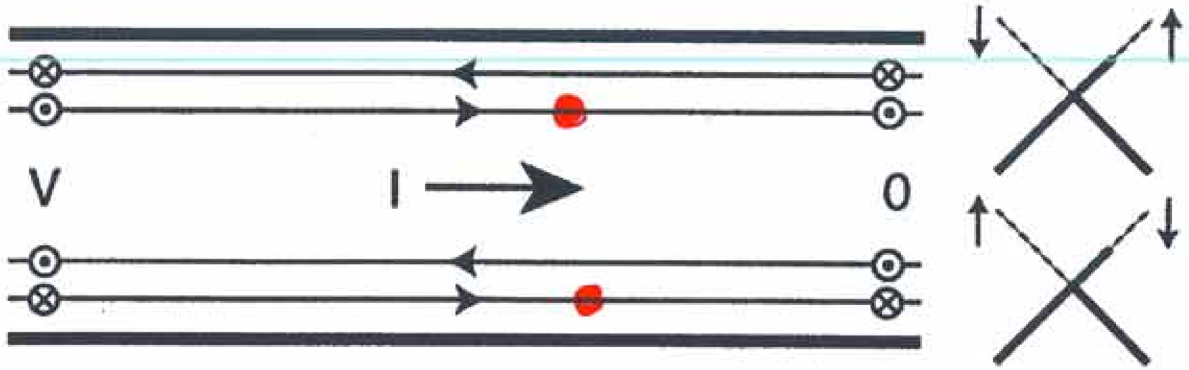
Think dynamically!

(N.B. Electrons move opposite to currents [Franklin])



15.18

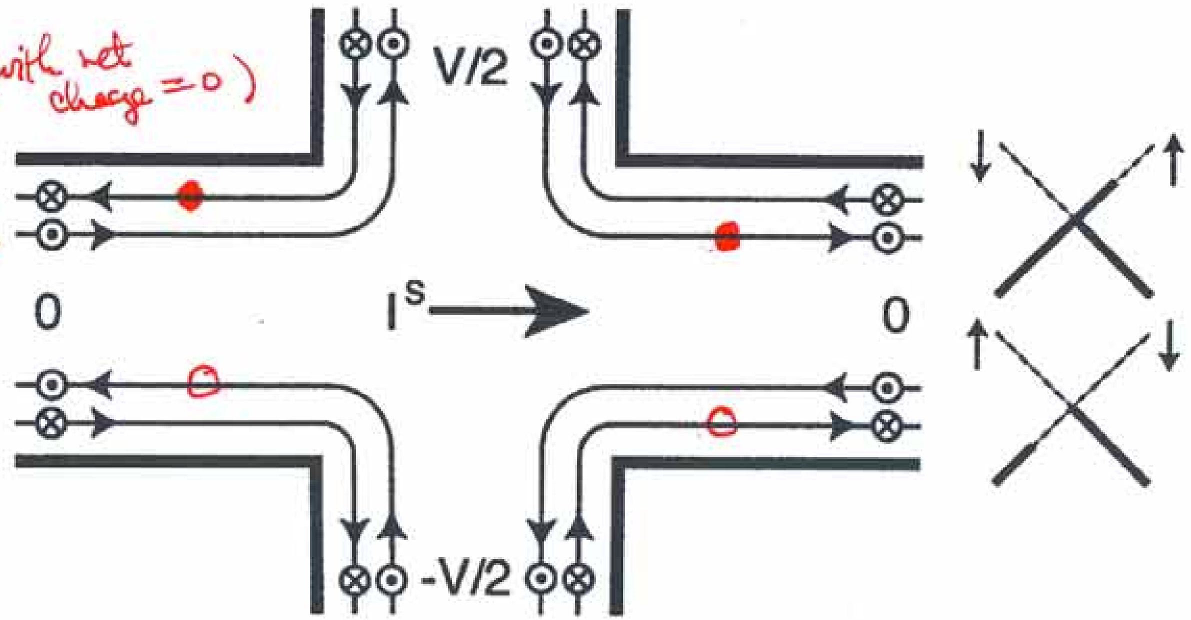
Net spin accumulates at edges!



(a)

(with net charge = 0)

Injected spin separates charge & ~~[illegible]~~ induces voltage!



$\boxed{15.20}$

Scholium to massless Dirac equation (conjecture)

Emerged by calculation from nearest-neighbor hopping.

Mass (opening of gap) for $\begin{cases} \text{A vs. B potential} \\ t_2 \sin\phi \neq 0 \quad (\text{T violation}) \end{cases}$

Conjecture: consequence of C_{6v} + T

2d rep. of C_{3v}

$E \to -E$ fr $C_{6v}/C_{3v} \times T$

This, and transport minimum, need clarification.

↳ [Ziegler PRL 80, 3113]

15.3 Z_2 Invariant for Quantum Spin Hall

Kane + Mele cond-mat/0506581

15.3.1 As we've discussed, the critical issue is whether there's an odd number of Kramers pairs within the bulk gap.

There's a Z_2 topological invariant for T-invariant systems, defined directly in terms of Bloch Hamiltonians, that answers this question.

(N.B. This is over and above the TTKN integers (Chern #s), which do not require - in fact they trivialize for - T-invariance)

15.3.2 On background, for graphene (Haldane's model)

$$H = t \sum_{\langle ij \rangle} c_i^\dagger c_j + i \Delta_{SO} \sum_{\langle\langle ij \rangle\rangle} \nu_{ij} c_i^\dagger S^z c_j$$

N.N. hopping

NNN, oriented "spin-split"

$$+ i \lambda_R \sum_{\langle ij \rangle} c_i^\dagger (S \times \hat{d}_{ij})_z c_j + \lambda_v \sum_i \varepsilon_i c_i^\dagger c_i$$

Rashba

A↔B breaking

the Bloch Hamiltonian is

$$\mathcal{H}(\vec{k}) = \sum_{a=1}^{5} d_a(\vec{k})\,\Gamma^a + \sum_{a<b=1}^{5} d_{ab}(\vec{k})\,\Gamma^{ab}$$

(4 component spinors : 2 for A or B and two for spin
$\vec{\sigma}$ … $\vec{s}$)

$$\Gamma^{(1,2,3,4,5)} = (\sigma^x \otimes \mathbb{1},\ \sigma^z \otimes \mathbb{1},\ \sigma^y \otimes s^x,\ \sigma^y \otimes s^y,\ \sigma^z \otimes s^z)$$

$$\Gamma^{ab} = \frac{1}{2i}[\Gamma^a, \Gamma^b]$$

Under time reversal

$$\Theta |u\rangle \equiv i(I \otimes s^y)|u\rangle^*$$

$$\Theta \Gamma^a \Theta^{-1} = \Gamma^a \quad ; \quad \Theta \Gamma^{ab} \Theta^{-1} = -\Gamma^{ab}$$

$\mathcal{H}$ is T symmetric $\iff$ $d_a(\vec{k}) = d_a(-\vec{k})$ and $d_{ab}(\vec{k}) = -d_{ab}(-\vec{k})$.

For Haldane's model :

d_1	$t(1+2\cos x \cos y)$	d_{12}	$-2t\cos x \sin y$
d_2	λ_v	d_{15}	$\Delta_{so}(2\sin 2x - 4\sin x \cos y)$
d_3	$\lambda_R(1-\cos x \cos y)$	d_{23}	$-\lambda_R \cos x \sin y$
d_4	$-\sqrt{3}\lambda_R \sin x \sin y$	d_{24}	$\sqrt{3}\lambda_R \sin x \cos y$
d_5	0	others	0

with $x \equiv k_x a/2$
$y \equiv \sqrt{3} k_y a/2$

15.23

15.3.3 Consider

$$P(\vec{k}) = \mathrm{Pf}\,[\langle u_i(\vec{k})|\Theta|u_j(\vec{k})\rangle]$$

(i.e. $\langle u_i(\vec{k})|\Theta|u_j(\vec{k})\rangle = \epsilon_{ij} P(\vec{k})$).

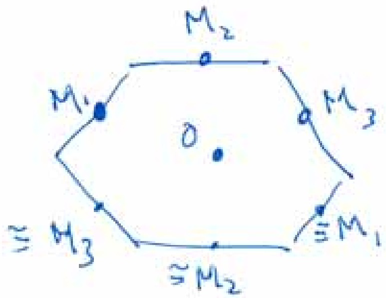
[$\langle u_i(\vec{k})|\Theta^3|u_j(\vec{k})\rangle \underset{\text{anti-unitarity}}{=} \langle u_j(\vec{k})|\Theta|u_i(\vec{k})\rangle$

$= -\langle u_i(\vec{k})|\Theta|u_j(\vec{k})\rangle$]

Generically this has isolated zeros (2 parameters, 1 real + 1 complex condition).

The zeros appear in pairs at momenta $\pm\vec{k}_i^*$, with opposite vorticity.

15.3.4 A single pair might annihilate if $+\vec{k}_i \cong -\vec{k}_i$ (i.e., up to a reciprocal lattice vector). That $\cong$ occurs at $\vec{k}_i = 0$ or one of the M points:



[canonical equivalence takes $\varphi_1 \to \varphi_2^*$, $\varphi_2 \to -\varphi_1^*$; others are U(2) rotations of this. and $\mathrm{Pf}(UM) = \det U\,\mathrm{Pf}\,M$]

But for a T-invariant system the eigenspaces at these points are equivalent (since $\Theta H_{\vec{k}}\Theta^{-1} = H_{-\vec{k}}$) so $|P(\vec{k})| = 1$.

~~canonical equivalence~~

15.24

15.3.5 Of course, nothing prohibits pairs of pairs (with opposite vorticity) from annihilating. Thus the # of pairs is conserved (against arbitrary T-invariant perturbations) mod 2.

15.3.6 It can be calculated by integrating the phase of P around 1/2 a Brillouin zone!

15.3.7 For $\lambda_v = 0$ things are not generic — $P(\vec{k})$ is real, and vanishes along lines. However, a regulated integral (inserting λ_v!) still defines the ~~[illegible]~~ Z_2 invariant.

15.3.8 The direct connection to transport and to many-body physics should be made explicit. [In the literature, it is heuristic at best.]