11.1 Pfaffians

11.1.1 Let \( M \) be an antisymmetric matrix of even order. The Pfaffian of \( M \) is defined to be

\[
\text{Pf} \; M = \sum \text{sgn}(\pi) \; M_{\pi(1)} M_{\pi(2)} \cdots M_{\pi(N)} \; \text{pairings}
\]

where \( \pi \) is the permutation necessary to get from 1, \ldots, 2N to \( abcd \cdots uv \) and \( \text{sgn} \; \pi = \pm 1 \) if \( \pi \) is \( \text{even} \) \( \frac{\text{odd}}{} \). This is well-defined, i.e., it depends only on how you pick the pairs.

Example: \((0 \ M_{12} \ -M_{12} \ 0)\) \[\text{Pf} = M_{12}\]

4x4: \[\text{Pf} = M_{12} M_{34} - M_{13} M_{24} + M_{14} M_{23}\]

11.1.2 The key mathematical properties of the Pfaffian are:

a) It's antisymmetric in all indices (must examine paired and unpaired cases separately)

b) \( (\text{Pf} \; M) = \det M \)

c) \( \text{Pf}(WMW^T) = \det W \; \text{Pf} \; M \)
Proof of a): paired case: $M_{ab} = -M_{ba}$

unpaired case: terms with $M_{xM_{y}}$ and $M_{yM_{x}}$ occur with opposite parity $\alpha$.

Proof of b): go to canonical form

\[
\begin{pmatrix}
0 & M_{12}
\end{pmatrix}
\begin{pmatrix}
0 & M_{23}
\end{pmatrix}
\begin{pmatrix}
0 & M_{34}
\end{pmatrix}
\begin{pmatrix}
0 & M_{45}
\end{pmatrix}
\begin{pmatrix}
0 & M_{56}
\end{pmatrix}
\begin{pmatrix}
0 & M_{61}
\end{pmatrix}
\end{pmatrix}
\]

orthogonal by transformation, + invoke c)

Alternatively, note degree is minimal (for both) given vanishing when any 2 columns are equal.

Proof of c): Look what happens to a typical term.

$M_{12} M_{34} \rightarrow a_1 a_4 a_2 a_3 M_{ij} M_{kl}$

Totally antisymmetrize on bottom indices $\Rightarrow$ take out determinant, reconstruct Pfaffian.

Put another way: plug into $\varepsilon_{ijkl} M_{ij} M_{kl}$... form.
The Pfaffian appears in the real-space form of BCS wavefunctions.

The momentum-space form pairs definite modes \((kt + -kt)\), so you get a product form as in \( \text{Pf} \left( \begin{pmatrix} 1 \, M_{2z} \\ -M_{2z} \, 0 \end{pmatrix} \right) \).

Transforming, you get the Pfaffian according to c) above.

11.2 Pfaffian QHE states

11.2.1 Motivations

a) The flux-statistics trading procedure \( \frac{\partial \Theta}{\partial n} = \frac{1}{2} \) relates fermion at \( \nu = \frac{1}{2} \) to fermion at \( \nu = \infty \), i.e. in magnetic field. These are, generically, gapped by Cooper pairing.

b) Putting in multi-zeros, a la Laughlin, might override it, leaving a residual attractive interaction. We can compensate by pairing with inverse powers of \( z_i - z_j \).
11.2.2 Ground state trial wave function for $\nu = \frac{1}{2}$

$$\Psi_{1/2}(z_1, z_2, ..., z_{2n}) = Pf \frac{1}{z_i - z_j} \prod (z_i - z_j)^2 e^{-\frac{1}{2}\sum(1/z_i^2)}$$

Pairing in Laughlin bosonic form
Pairing (i.e. $l = -1$)

Note: Antisymmetric
Non-singular