Lecture 11: Quantum Hall Effect II: Pfaffian State

11.1 Pfaffians

11.1.1 ~~Let~~ Let M be an antisymmetric matrix of even order. The Pfaffian of M is defined to be

$$\mathrm{Pf}\, M = \sum_{\text{pairings}} \mathrm{sgn}(\pi)\, M_{ab} M_{cd} \cdots M_{uv} \propto \epsilon^{ijkl\ldots} M_{ij} M_{kl} M_{\ldots}$$

where π is the permutation necessary to get from $1, \ldots 2N$ to $abcd \ldots uv$ and $\mathrm{sgn}\,\pi = \pm 1$ if π is $\begin{cases}\text{even}\\ \text{odd}\end{cases}$. This is well-defined, i.e. it depends only on how you pick the pairs.

example: $\begin{pmatrix} 0 & M_{12} \\ -M_{12} & 0 \end{pmatrix}$ $\quad \mathrm{Pf} = M_{12}$

$$4\times 4: \quad \mathrm{Pf} = M_{12} M_{34} - M_{13} M_{24} + M_{14} M_{23}$$

11.1.2 The key mathematical properties of the Pfaffian are:

a) It's antisymmetric in all indices (~~Proof: examine paired + unpaired cases separately~~)

b) $(\mathrm{Pf}\, M)^2 = \det M$

c) $\mathrm{Pf}(WMW^t) = \det W\; \mathrm{Pf}\, M$

~~$M_{ab} = -M_{ab}$~~

~~sgn~~ $M_{ax} M_{by}$

Proof of a):
paired case: $M_{ab} = -M_{ba}$
unpaired case: terms with $M_{ax} M_{by}$ and $M_{ay} M_{bx}$ occur
with opposite parity π.

Proof of b): go to canonical form
$$\begin{pmatrix} \begin{pmatrix} 0 & M_{12} \\ -M_{12} & 0 \end{pmatrix} & & \\ & \begin{pmatrix} 0 & M_{34} \\ -M_{34} & 0 \end{pmatrix} & 0 \\ 0 & & \ddots \end{pmatrix}$$

by ~~canonical~~ orthogonal transformation, + invoke c)

Alternatively, note degree is minimal (for both) given vanishing when any 2 columns are =.

Proof of c): Look what happens to a typical term.

$$M_{12} M_{34} \to a_1^i a_2^j a_3^k a_4^l M_{ij} M_{kl}$$

Totally antisymmetrize on bottom indices $\Rightarrow$ take out determinant, reconstruct Pfaffian.

Put another way: plug into $\epsilon^{ijkl\cdots} M_{ij} M_{kl} \cdots$ form.

11.1.3 The Pfaffian appears in the real-space form of BCS wavefunctions

The momentum-space form pairs definite modes $(k\uparrow \leftrightarrow -k\downarrow)$, so you get a product form as in $\mathrm{Pf}\begin{pmatrix} \begin{pmatrix} 0 & M_{12} \\ -M_{12} & 0 \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} 0 & M_{2N-1,2N} \\ -M_{2N-1,2N} & 0 \end{pmatrix} \end{pmatrix}$.

Transforming, you get the Pfaffian according to c) above.

11.2 Pfaffian QHE states

11.2.1 Motivations

a) The flux-statistics trading procedure $\Delta \frac{\theta}{\pi} = \Delta \frac{1}{\nu}$ relates ~~ferm~~ fermions at $\nu = \frac{1}{2}$ to fermions at $\nu = \infty$, i.e. in zero magnetic field. These are, generically, gapped by Cooper pairing.

b) Putting in ~~multi~~ multi-zeros, a lá Laughlin, might overdo it, leaving a residual attractive interaction. We can compensate by pairing with inverse powers of $z_i - z_j$.

11.2.2 Ground state trial wavefunction for $\nu = \frac{1}{2}$

$$\Psi_{1/2}(z_1, \ldots, z_{2N}) = \underbrace{\mathrm{Pf}\,\frac{1}{z_i - z_j}}_{\text{Pairing in } p_x + i p_y \text{ (i.e. } l=-1)}\ \underbrace{\prod (z_i - z_j)^2\ e^{-\frac{1}{4\ell_B^2}\sum |z_i|^2}}_{\text{Laughlin bosonic form}}$$

Note: Antisymmetric ✓

Non-singular