14.3 Relativistic Hall Effect by Mode Analysis

14.3.1 For this section we will fix conventions as

\[ H = \alpha_i (i \alpha \cdot q \Lambda_i) + \beta m + \phi \]

\[ \alpha_i = \beta_0 \alpha_i \quad \beta = \beta_0 \]
\[ \alpha_i^2 = \beta = 1; \{\alpha_i, \alpha_j\} = 0 = \{\beta, \alpha_i\} \]

\( \phi \) is the electric potential, which we'll use only later (14.3.1).

14.3.2 We saw that this describes the low-energy modes of graphene near half filling. \( m = 0 \) in "real" graphene; and there are two copies, each described by a 2-component spinor. With the Haldane terms turned on, (violating T and P) there are effective masses opposed at the two relevant "Dirac points" \( \pm k^* \). The momenta in the effective Dirac equation refer of course to small deviations from \( \pm k^* \), i.e. slow spatial modulations of these plane waves.

14.3.3 When necessary for concreteness we will take \( \beta_0 = \frac{1}{3}, \beta_1 = i \beta_i, \beta_2 = i \beta_2. \)
We consider applying a magnetic field. We consider a gauge: \( A_y = B x \) \((+A_z = 0)\).

With \( \psi = e^{i k \cdot \phi} \), the Schrödinger–Dirac equation \( H \psi = E \psi \)

becomes

\[
E \psi = \left[ \alpha_x (k - q B y) + \alpha_y (i \partial_y) + \beta m \right] \psi = 0 \psi
\]

Applying \( \psi \) again, and using the properties of \( \alpha + \beta \), we find

\[
E^2 \psi = \left[ (k - q B y) \partial_y - m^2 + \alpha_y \partial_x \right. \\
\left. \left. (i \partial_y - q B y) + \alpha_x \partial_y (-q B y) \right] \psi
\]

\[
= \left[ (k - q B y)^2 - \partial_y^2 - m^2 - i q B \partial_y \alpha_x \right] \psi
\]

\[
= \left[ (k - q B y)^2 - \partial_y^2 - m^2 - q B \partial_z \right] \psi
\]
Letting \( \sigma_3 \phi_+ = \phi_+ \) (upper component!)
\( \sigma_3 \phi_- = -\phi_- \)

we have
\[
E^2 \phi_+ = \left( (k-qBy)^2 - \delta_y^2 + m^2 + qB \right) \phi_+
\]

Apart from a shift in \( y \) by \( \frac{k}{qB} \) and the additive factors \( m^2 + qB \),

we recognize (and the appearance of \( E^2 \) rather than \( E! \)) we recognize this as the Schrödinger equation for a nonrelativistic harmonic oscillator with \( \frac{1}{2m} = \frac{1}{1} \) and spring constant \( K/2 = (qB)^2 \). Thus for \( \kappa \), not \( k \)

the frequency we have \( \omega^2 = K/\hbar = 4(qB)^2 \)

and for the eigenvalues
\[
E^2 = 2nqB + n21qB^2 + m^2 + qB
\]

\( n > 0 \) here!

14.3.3 From the solutions \( \sim \phi = E\phi \) we get solutions of the original equation \( \sim \phi = E\phi \) using \( \sim \phi = \phi \pm \frac{\phi}{E} \)

\[
\left[ \sim \phi = \phi \pm \frac{\phi}{E} = \phi \pm E\phi = \pm E(\phi \pm \frac{\phi}{E}) = \pm E\phi \right]
\]

[Note: \( E^2 \) for \( \phi_+ \) with \( n \) matches \( E^2 \) for \( \phi_- \) with \( n-1, \ n \geq 1 \).]
14.3.4 However this won't hold if $E=0$; in that case we must go back to the original equation.

We now assume $qB > 0$ for concreteness.

$E=0$ solution exists for $\phi_+$ with $n=0$ (and $\bar{n}=0$).

Our equation is

$$0 = \left[ \alpha_y (k-qB) + \omega_y (i\dot{\omega}_y) \right] \phi_+$$

or

$$0 = \left[ \left( k-qB \right) + 2\alpha \omega_y (i\dot{\omega}_y) \right] \phi_+$$

solved by $\phi_+(y) \propto e^{-qB} (y - \frac{k}{qB})$. N.B. non-discrete ($qB > 0$).

14.3.5 If $qB < 0$ the roles of $\phi_+$ and $\phi_-$ are reversed, ...

14.3.6 Adding $\omega_m$ to $0$ in 14.3.4, we see that the 0 mode $\bar{\nu} \rightarrow m$ for $\phi_+$

is shifted to positive energy (unoccupied) for $m>0$, negative energy (occupied) for $m<0$. 

\[ \text{\[} \] \]
14.3.7 When the $m$, at the two critical momenta, have the same sign, the occupancy conditions agree. Thus a small magnetic field will induce a change density - the two modes get "devered". Opposite $m$, occur for sufficiently large values of $B$. In the Halderen model.

[N.B.: Halderen uses a parity-reversing convention at the two critical momenta, so he says they have "opposite" signs whereas I say "same", and vice-versa.]

14.3.8 When a small $B$ induces change, there must be transverse current flowing in to supply it.

Thus there is a Hall effect (transverse conductivity).

This is the philosophy of the "Streda formula".
It also follows from the covariant relation $j = E_{\omega} r_{\mu\nu} F^{\mu\nu}$. 

\[ B \quad dB > 0 \]