The first part of this course will cover the foundational material of homogeneous big bang cosmology. There are three basic topics:

1) General Relativity
2) Cosmological Models with Idealized Matter
3) Cosmological Models with Understood Matter

1) General Relativity

references: Landau-Lifshitz 2
Weinberg
(Misner-Thorne-Wheeler)

This will be a terse introduction to GR.

It will be logically complete, ~~but~~ and adequate for our later purposes, but a lot of good stuff is left out (astrophysical applications, tests, black holes, gravitational radiation ...)

1.1 Transformations and Metric

We want equations that are independent of coordinates. More precisely, we want them to be invariant under ("smooth") reparameterizations

$$x'^{\mu} = x'^{\mu}(x)$$

To do local physics we need derivatives. Of course

$$dx'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} dx^{\nu}$$ (note: summation convention)

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} \equiv R^{\mu}{}_{\nu} \in GL(4)$$ (invertible real matrix)

We want to have special relativity in small empty regions, so we must introduce more structure. The right thing is to introduce a symmetric, non-singular (signature $+ - - -$) tensor field $g_{\mu\nu}(x)$ so that we can define intervals

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu \qquad (\text{"Pythagoras"})$$

For this to be invariant $(g'_{\mu\nu}\, dx'^\mu\, dx'^\nu = g_{\mu\nu}\, dx^\mu\, dx^\nu)$ we need

$$g'_{\mu\nu}(x') = (R^{-1})^{\alpha}_{\mu}\,(R^{-1})^{\beta}_{\nu}\; g_{\mu\nu}(x)$$

Since $R^{\mu}_{\ \nu} = \dfrac{\partial x'^\mu}{\partial x^\nu}$, $R^{-1\,\rho}_{\ \ \sigma} = \dfrac{\partial x^\rho}{\partial x'^\sigma}$

(chain rule: $\delta^\alpha_\beta = \dfrac{\partial x'^\alpha}{\partial x'^\beta} = \dfrac{\partial x^\lambda}{\partial x'^\beta}\dfrac{\partial x'^\alpha}{\partial x^\lambda} \ldots$)

We can write the transformation law for $g_{\mu\nu}$ in matrix form

$$G' = R^{-1}\, G\, (R^{-1})^T$$

From linear algebra we can insure G' is diagonal with ± 1 (or 0) entries. The signature, e.g. $\begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$, is determined

There are residual transformations that leave this form of $g_{\mu\nu}$ intact. They are the Lorentz transformations!

Generalizing $g_{\mu\nu}$, dx^μ we define tensors of more general kinds

$$T_{\mu_1 \cdots \mu_m}{}^{\nu_1 \cdots \nu_n}(x)$$

by the transformation law

$$T'_{\mu_1 \cdots \mu_m}{}^{\nu_1 \cdots \nu_n}(x') = (R^{-1})^{\alpha_1}_{\mu_1} \cdots (R^{-1})^{\alpha_m}_{\mu_m} R^{\nu_1}_{\beta_1} \cdots R^{\nu_n}_{\beta_n} \times T_{\alpha_1 \cdots \alpha_m}{}^{\beta_1 \cdots \beta_n}(x)$$

<u>ex</u>: Inverse metric $g^{\mu\nu}$

$$g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu$$

Operations:

Multiplication by numbers.

Tensors of the same type can be added

Outer product

$$V^{\nu_1} W^{\nu_2}{}_{\mu} \equiv T^{\nu_1 \nu_2}{}_{\mu}$$

Contraction

$$T_{\mu_1 \mu_2 \cdots \mu_m}{}^{\mu_1 \nu_2 \cdots \nu_n} = \tilde{T}_{\mu_2 \cdots \mu_m}{}^{\nu_2 \cdots \nu_n}$$

ex: $g_{\mu\nu}\, dx^\alpha\, dx^\beta = T_{\mu\nu}{}^{\alpha\beta}$ a tensor

contraction $T_{\mu\nu}{}^{\mu\nu} = ds^2$

"0" is a tensor (all components = 0).

δ^μ_ν is a tensor

1.2. Covariant Derivative: Affine

Scalar field $\varphi'(x') = \varphi(x)$

Vector $A'_\mu(x') = \dfrac{\partial x^\alpha}{\partial x'^\mu} A_\alpha(x)$ $\left(= (R^{-1})^\alpha_\mu A_\alpha \right)$

Operator $\partial'_\nu \equiv \dfrac{\partial}{\partial x'^\nu} = \dfrac{\partial x^\alpha}{\partial x'^\nu} \dfrac{\partial}{\partial x^\alpha}$

Invariant derivative?

$$\partial'_\nu A'_\mu = \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial}{\partial x^\alpha}\left(\frac{\partial x^\beta}{\partial x'^\mu} A_\beta\right)$$

$$= \underbrace{\frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\mu}\partial_\alpha A_\beta}_{\text{good}} + \underbrace{\frac{\partial^2 x^\beta}{\partial x'^\nu \partial x'^\mu} A_\beta}_{\text{bad}}$$

(hard to use)
- not a tensor

Add correction term: $\nabla_\nu A_\mu \equiv \partial_\nu A_\mu - \Gamma^\lambda_{\nu\mu} A_\lambda$

$$\nabla'_\nu A'_\mu = S^\alpha_\nu S^\beta_\mu \partial_\alpha A_\beta + \frac{\partial^2 x^\sigma}{\partial x'^\mu \partial x'^\nu} A_\sigma - \Gamma'^\lambda_{\nu\mu} S^\sigma_\lambda A_\sigma$$

where $\left[S^\alpha_\nu \equiv (R^{-1})^\alpha_\nu = \frac{\partial x^\alpha}{\partial x'^\nu} \right]$

$$\stackrel{?}{=} S^\alpha_\nu S^\beta_\mu \left(\partial_\alpha A_\beta - \Gamma^\sigma_{\alpha\beta} A_\sigma \right)$$

This will work if

$$\Gamma'^{\lambda}_{\nu\mu} \, S^{\sigma}_{\lambda} = S^{\alpha}_{\nu} S^{\beta}_{\mu} \Gamma^{\sigma}_{\alpha\beta} + \frac{\partial^2 x^\sigma}{\partial x'^\nu \partial x'^\mu}$$

or

$$\Gamma'^{\lambda}_{\nu\mu} = R^{\lambda}_{\sigma} S^{\alpha}_{\nu} S^{\beta}_{\mu} \Gamma^{\sigma}_{\alpha\beta} + \frac{\partial x'^\lambda}{\partial x^\sigma} \frac{\partial^2 x^\sigma}{\partial x'^\nu \partial x'^\mu}$$

Note that the inhomogeneous part is symmetric in $\mu \leftrightarrow \nu$. So we can assume $\Gamma^{\lambda}_{\alpha\beta} = \Gamma^{\lambda}_{\beta\alpha}$ consistently. (The antisymmetric "torsion" part is a <u>tensor</u> on its own!)

Given Γ we can take covariant derivatives of arbitrary tensors as

$$\nabla_\alpha T_{\mu_1 \dots \mu_m}{}^{\nu_1 \dots \nu_n} = \partial_\alpha T_{\mu_1 \dots \mu_m}{}^{\nu_1 \dots \nu_n} - \Gamma^{\lambda}_{\alpha\mu_1} T_{\lambda\mu_2 \dots \mu_m}{}^{\nu_1 \dots \nu_n} - \dots - \Gamma^{\lambda}_{\alpha\mu_m} T_{\mu_1 \dots \lambda}{}^{\nu_1 \dots \nu_n} + \Gamma^{\nu_1}_{\alpha\lambda} T_{\mu_1 \dots \mu_m}{}^{\lambda\nu_2 \dots \nu_n} + \dots$$

This gives a [illegible] Leibniz rule for products, etc.

1.3 Covariant Derivative: Metric

Big result: given metric, there is a unique preferred connection Γ

Demand $\nabla_\lambda g_{\mu\nu} = 0$ (and symmetry)

$$0 = \partial_\lambda g_{\mu\nu} - \Gamma^\alpha_{\lambda\mu} g_{\alpha\nu} - \Gamma^\alpha_{\lambda\nu} g_{\alpha\mu}$$

$$0 = \partial_\mu g_{\lambda\nu} - \Gamma_{\lambda\mu} g_{\alpha\nu} - \Gamma^\alpha_{\mu\nu} g_{\alpha\lambda}$$

$$0 = \partial_\nu g_{\lambda\mu} - \Gamma^\alpha_{\lambda\nu} g_{\alpha\mu} - \Gamma^\alpha_{\mu\nu} g_{\alpha\lambda}$$

Subtract 1st line from sum of 2nd + 3rd:

$$2\Gamma^\alpha_{\mu\nu} g_{\alpha\lambda} = \partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}$$

$\times \frac{g^{\lambda\beta}}{2}$: $$\Gamma^\beta_{\mu\nu} = \frac{1}{2} g^{\lambda\beta} (\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})$$

Conversely, this works!

Extra term in Γ':

$$\frac{1}{2} g^{\lambda\beta}\Big[\left(\partial'_\mu \frac{\partial x^\sigma}{\partial x'^\lambda}\right) g_{\sigma\nu} + \left(\partial'_\nu \frac{\partial x^\sigma}{\partial x'^\lambda}\right) g_{\sigma\mu} + \boxed{\left(\partial'_\mu \frac{\partial x^\sigma}{\partial x'^\nu}\right) g_{\lambda\sigma} + \left(\partial'_\nu \frac{\partial x^\sigma}{\partial x'^\mu}\right) g_{\lambda\sigma}} - \left(\partial'_\lambda \frac{\partial x^\sigma}{\partial x'^\mu}\right) g_{\sigma\nu} - \left(\partial'_\lambda \frac{\partial x^\sigma}{\partial x'^\nu}\right) g_{\sigma\mu} \Big]$$

The boxed terms give the desired inhomogeneous terms; the others cancel.

1.4 Measure

$$d^4x' = \underbrace{\det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right)}_{\text{Jacobian}} d^4x \qquad ; \quad = \det R$$

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$$

$$G' = R^{-1} G (R^{-1})^T$$

$$\det g'_{\mu\nu} = \frac{1}{(\det R)^2} \det g_{\mu\nu}$$

Write $g \equiv \det(g_{\mu\nu})$; then

$$\sqrt{g'}\, d^4x' = \sqrt{g}\, d^4x$$

is invariant measure.

1.5 Curvature

For dynamics of $g_{\mu\nu}$, want it to appear with derivatives. Tensors?

Trick: $(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) A_\beta \equiv - \underbrace{R^\alpha{}_{\beta\mu\nu}}_{\text{involves } g} \underbrace{A_\alpha}_{\text{no derivatives}}$

Thus $R^\alpha{}_{\beta\mu\nu}$ automatically transforms as a proper tensor; the "hard" part is to show that the derivatives on A_α all cancel, so we get this form

$$\nabla_\mu A_\beta = \partial_\nu A_\beta - \Gamma^\lambda_{\nu\beta} A_\lambda$$

$$\nabla_\mu(\nabla_\nu A_\beta) = \partial_\mu (\nabla_\nu A_\beta) - \underbrace{\cancel{\Gamma^\sigma_{\mu\nu} \nabla_\sigma A_\beta}}_{\text{symmetric} \Rightarrow \text{drop it}} - \Gamma^\sigma_{\mu\beta} \nabla_\nu A_\sigma$$

$$= \cancel{\partial_\mu \partial_\nu A_\beta} - \partial_\mu (\Gamma^\sigma_{\nu\lambda} A_\sigma) - \Gamma^\sigma_{\mu\nu} \partial_\nu A_\sigma$$

$$+ \Gamma^\rho_{\mu\beta} \Gamma^\sigma_{\nu\rho} A_\sigma$$

$$= - \partial_\mu \Gamma^\sigma_{\nu\lambda} A_\sigma - \cancel{\Gamma^\sigma_{\nu\lambda} \partial_\mu A_\sigma} - \Gamma^\sigma_{\mu\lambda} \partial_\nu A_\sigma$$

$$+ \Gamma^\rho_{\mu\beta} \Gamma^\sigma_{\nu\rho} A_\sigma$$

so: $$R^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\nu\beta} - \partial_\nu \Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\mu\rho} \Gamma^\rho_{\nu\beta} - \Gamma^\alpha_{\nu\rho} \Gamma^\rho_{\mu\beta}$$

Symmetry properties of $R_{\alpha\beta\gamma\delta}$

We can go to a frame where $\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = 0$ (at point of interest). These are "geodesic coordinates". Then $\Gamma^\alpha_{\beta\gamma} = 0$ (but not its derivatives!)

Then ~~after some~~

$$R^\alpha{}_{\beta\mu\nu} = \frac{1}{2}\Big[\partial_\mu \big(g^{\alpha\sigma}(\partial_\nu g_{\sigma\beta} - \partial_\beta g_{\nu\sigma} - \partial_\sigma g_{\beta\nu}\big) - (\mu \leftrightarrow \nu)\Big]$$

$$= \frac{1}{2} g^{\alpha\sigma}\left[\partial_\mu\partial_\beta g_{\nu\sigma} - \partial_\mu\partial_\sigma g_{\beta\nu} - \partial_\nu\partial_\beta g_{\mu\sigma} + \partial_\nu\partial_\sigma g_{\beta\mu}\right]$$

$$= g^{\alpha\sigma} R_{\sigma\beta\mu\nu}$$

with

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left[\partial_\alpha\partial_\delta g_{\beta\gamma} + \partial_\beta\partial_\gamma g_{\alpha\delta} - \partial_\alpha\partial_\gamma g_{\beta\delta} - \partial_\beta\partial_\delta g_{\alpha\gamma}\right]$$

$$\Longrightarrow \quad R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}$$

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$$

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0$$

(e.g., look at coeff. of $\partial_\alpha \partial_\delta g_{\beta\gamma}$:

$+1$ in $R_{\alpha\beta\gamma\delta}$

-1 in $R_{\alpha\gamma\delta\beta}$)

0 in $R_{\alpha\delta\beta\gamma}$

Since these are tensor identities, they hold in any frame!

Also notable is the Bianchi identity

$$\nabla_\alpha R_{\mu\nu\beta\gamma} + \nabla_\beta R_{\mu\nu\gamma\alpha} + \nabla_\gamma R_{\mu\nu\alpha\beta} = 0$$

It follows from

$$[\nabla_\alpha, [\nabla_\beta, \nabla_\gamma]] + [\nabla_\beta [\nabla_\gamma, \nabla_\alpha]] + [\nabla_\gamma [\nabla_\alpha \nabla_\beta]] = 0$$

$$\begin{aligned}&\nabla_\alpha \nabla_\beta \nabla_\gamma - \nabla_\alpha \nabla_\gamma \nabla_\beta - \nabla_\beta \nabla_\gamma \nabla_\alpha + \nabla_\gamma \nabla_\beta \nabla_\alpha \\ &+ \nabla_\beta \nabla_\gamma \nabla_\alpha - \nabla_\beta \nabla_\alpha \nabla_\gamma - \nabla_\gamma \nabla_\alpha \nabla_\beta + \nabla_\alpha \nabla_\gamma \nabla_\beta \\ &+ \nabla_\gamma \nabla_\alpha \nabla_\beta - \nabla_\gamma \nabla_\beta \nabla_\alpha - \nabla_\alpha \nabla_\beta \nabla_\gamma + \nabla_\beta \nabla_\alpha \nabla_\gamma = 0.\end{aligned}$$

e.g. $\nabla_\alpha [\nabla_\beta \nabla_\gamma] A_\mu = -\nabla_\alpha (R^\nu{}_{\mu\beta\gamma} A_\nu) = -\nabla_\alpha R^\nu{}_{\mu\beta\gamma} A_\nu - R^\nu{}_{\mu\beta\gamma} \nabla$ ①②

$-[\nabla_\beta \nabla_\gamma] \nabla_\alpha A_\mu = R^\nu{}_{\alpha\beta\gamma} \nabla_\nu A_\mu + R^\nu{}_{\mu\beta\gamma} \nabla_\alpha A_\nu$ ③④

② cancels against ④, ③ will go away by the symmetry of $R^{\nu}{}_{\alpha\beta\gamma} + R^{\nu}{}_{\gamma\alpha\beta} + R^{\nu}{}_{\beta\gamma\alpha} = 0$,

so ① ~~gives~~ generates the Bianchi identity.

This identity is the gravity analogue of $\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$ in electromagnetism, or $\vec{\nabla}\cdot\vec{B} = 0$ & $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (existence of vector potential)

1.6 Invariant Actions

Since we have an invariant measure $\int \sqrt{g}\, d^4x\, (\mathcal{L})$, we get invariant field theories by putting invariant expressions inside $(\mathcal{L})$.

Given a special-relativistic invariant theory, we just need to change

$$d^4x \to \sqrt{g}\, d^4x$$

$$\eta_{\mu\nu} \to g_{\mu\nu}$$

$$\partial_\mu \to \nabla_\mu$$

to make a general-relativistic invariant theory.

This is the "minimal coupling" procedure.

examples : a) scalar field $\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi)$

b) ~~vector~~ transverse vector field

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \Gamma^\sigma_{\mu\nu} A_\sigma - \partial_\nu A_\mu + \Gamma^\sigma_{\nu\mu} A_\sigma$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu$$

(no need for Γ, or ∂g)

$$\mathcal{L} = -\frac{1}{4} g^{\alpha\gamma} g^{\beta\delta} F_{\alpha\beta} F_{\gamma\delta}$$

Supports gauge symmetry $A_\mu \to A_\mu + \partial_\mu \chi$

c) Longitudinal vector field (instructive)

$$\nabla_\mu A^\mu = \partial_\mu A^\mu + \Gamma^\mu_{\mu\alpha} A^\alpha$$

$$\Gamma^\mu_{\mu\alpha} = \frac{1}{2} g^{\mu\sigma} (\partial_\alpha g_{\sigma\mu} + \underbrace{\partial_\mu g_{\alpha\sigma} - \partial_\sigma g_{\mu\alpha}}_{\text{cancel}})$$

$$= \frac{1}{2} g^{\mu\sigma} \partial_\alpha g_{\sigma\mu} = \frac{1}{\sqrt{g}} \partial_\alpha \sqrt{g}$$

[To prove $\partial_\alpha g = g g^{\mu\sigma} \partial_\alpha g_{\mu\sigma}$,

use expansion by minors & expansion for inverse matrix. Check on diagonal matrices!]

So $\nabla_\mu A^\mu = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} A^\mu)$

Thus $\int d^4x \sqrt{g}\, \nabla_\mu A^\mu = \int d^4x\, \partial_\mu (\sqrt{g} A^\mu)$

is semi-trivial - a boundary term.

$\int d^4x \sqrt{g}\, \nabla_\mu A^\mu \nabla_\nu A^\nu$ gives dynamics.

Supports gauge transformation

$$\sqrt{g} A^\mu \to \sqrt{g} A^\mu + \epsilon^{\mu\nu\rho\sigma} \partial_\nu \Lambda_{\rho\sigma}$$

$$\Lambda_{\rho\sigma} = - \Lambda_{\sigma\rho}$$

d) gravity itself

$$R = g^{\beta\delta} R^{\alpha}{}_{\beta\gamma\delta} \qquad \text{and} \quad 1$$

are invariant. The latter is non-trivial, due to the measure factor $\int d^4x \sqrt{g}$. ["Cosmological term."]

e) spinors! – Foundational, but relegated to appendix

1.7 Field Equations

~~Plot~~

scalar field
$$S = \int d^4x \, \overbrace{\sqrt{g} \left(\tfrac{1}{2} g^{\mu\nu} \partial_\mu \varphi \, \partial_\nu \varphi - V(\varphi) \right)}^{\Lambda}$$

$$\partial_\mu \frac{\delta \Lambda}{\delta \partial_\mu \varphi} = \frac{\delta \Lambda}{\delta \varphi}$$

$$\partial_\mu \left(\sqrt{g} \, g^{\mu\nu} \partial_\nu \varphi \right) = - \sqrt{g} \, V'(\varphi)$$

or $$\nabla_\mu \left(g^{\mu\nu} \partial_\nu \varphi \right) = - V'(\varphi)$$

transverse vector

$$S = -\frac{1}{4}\int d^4x\ \sqrt{g}\, g^{\alpha\gamma} g^{\beta\delta} (\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial_\gamma A_\delta - \partial_\delta A_\gamma)$$
$$- \int d^4x\ \sqrt{g}\ j^\mu A_\mu$$

coupling to current

$$\partial_\mu \frac{\delta \sqrt{g}\mathcal{L}}{\delta \partial_\mu A_\nu} = -\partial_\mu \left(\sqrt{g}\, g^{\mu\gamma} g^{\nu\delta} (\partial_\gamma A_\delta - \partial_\delta A_\gamma)\right)$$
$$= -\partial_\mu (\sqrt{g} F^{\mu\nu}) \underset{\text{exercise!}}{\doteq} -\sqrt{g}\, \nabla_\mu F^{\mu\nu}$$

$$\frac{\delta \sqrt{g}\mathcal{L}}{\delta A_\nu} = -\sqrt{g}\, j^\nu$$

eq^n of motion: $\nabla_\mu F^{\mu\nu} = j^\nu$

or $\partial_\mu (\sqrt{g}\, g^{\mu\gamma} g^{\nu\delta} F_{\gamma\delta}) = \sqrt{g}\, j^\nu$

consistency: $\partial_\nu (\sqrt{g}\, j^\nu) = 0$

$$\shortparallel$$

$$\sqrt{g}\, \nabla_\nu j^\nu$$

<u>longitudinal vectr</u>

$$\int \sqrt{g}\, \nabla_\mu A^\mu \nabla_\nu A^\nu = \int \sqrt{g}\, \frac{1}{\sqrt{g}} \partial_\mu(\sqrt{g} A^\mu)\, \frac{1}{\sqrt{g}} \partial_\nu(\sqrt{g} A^\nu)$$

$$\equiv \int \frac{1}{\sqrt{g}} \partial_\mu \rho^\mu \partial_\nu \rho^\nu$$

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \rho^\nu} = 2 \partial_\mu \left(\frac{1}{\sqrt{g}} \delta^\mu_\nu \partial_\sigma \rho^\sigma\right) = 2 \partial_\nu \left(\frac{1}{\sqrt{g}} \partial_\sigma \rho^\sigma\right)$$

if no source, $\frac{1}{\sqrt{g}} \partial_\sigma \rho^\sigma = \lambda$ constant

$$\int \sqrt{g}\, \nabla_\mu A^\mu \nabla_\nu A^\nu \rightarrow \lambda^2 \int \sqrt{g}$$

"Dynamical" cosmological term

Field equation for gravity

a) Hard part: vary $\sqrt{g}\, g^{\alpha\beta} R_{\alpha\beta}$

main trick: $\sqrt{g}\, g^{\alpha\beta}\, \delta R_{\alpha\beta}$ = total derivative

To prove it relatively painlessly, adopt locally geodesic coordinates ($\partial_\alpha g_{\beta\gamma} = 0$).

then
$$g^{\alpha\beta}\, \delta R_{\alpha\beta} = g^{\alpha\beta} \left\{ \partial_\mu\, \delta \Gamma^{\mu}_{\alpha\beta} - \partial_\alpha\, \delta \Gamma^{\mu}_{\beta\mu} \right\}$$

$$= \partial_\mu \left(g^{\alpha\beta}\, \delta\Gamma^{\mu}_{\alpha\beta} - g^{\mu\beta}\, \delta \Gamma^{\alpha}_{\beta\alpha} \right)$$

(N.B. $\partial g = 0$)

$$\equiv \partial_\mu w^\mu$$

note that $\delta\Gamma$ has no inhomogeneous terms in transformation law – it is a tensor! So

$$g^{\alpha\beta}\, \delta R_{\alpha\beta} = \partial_\mu w^\mu \Rightarrow \nabla_\mu w^\mu$$

is now valid in any coordinate system.

So $\sqrt{g}\, g^{\alpha\beta}\, \delta R_{\alpha\beta} = \sqrt{g}\, \nabla_\mu w^\mu = \partial_\mu(\sqrt{g}\, w^\mu)$

is a boundary term; does not contribute to Euler-Lagrange equations.

Thus with $S = \kappa \int \sqrt{g}\, R$

we get

$$\delta S = \kappa \int \Big(\delta\sqrt{g}\, \overbrace{g^{\alpha\beta} R_{\alpha\beta}}^{R} + \sqrt{g}\, \delta g^{\alpha\beta} R_{\alpha\beta}\Big)$$

$$= \kappa \int -\tfrac{1}{2}\sqrt{g}\, \delta g^{\alpha\beta} R + \sqrt{g}\, R_{\alpha\beta}\, \delta g^{\alpha\beta}$$

$$= \kappa \int \sqrt{g}\,\big(R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R\big)\, \delta g^{\alpha\beta}$$

b) Cosmo. term

$$\delta S = -\Lambda \int \delta\sqrt{g} = +\Lambda \int \sqrt{g}\, \big(\tfrac{1}{2} g_{\alpha\beta}\big)\, \delta g^{\alpha\beta}$$

c) "Matter"

$$S = \int \sqrt{g}\,\Lambda$$

$$\delta S = \int \left(\frac{\delta \sqrt{g}\,\Lambda}{\delta g^{\alpha\beta}} - \partial_\mu \frac{\delta \sqrt{g}\,\Lambda}{\delta \partial_\mu g^{\alpha\beta}} \right) \delta g^{\alpha\beta}$$

We <u>define</u> the energy-momentum tensor by

$$\frac{\delta \sqrt{g}\,\Lambda}{\delta g^{\alpha\beta}} - \partial_\mu \frac{\delta \sqrt{g}\,\Lambda}{\delta \partial_\mu g^{\alpha\beta}} = \frac{\sqrt{g}}{2} T_{\alpha\beta}$$

See that this makes sense!

i) examples

ii) conservation law

i) Scalar field: $\Lambda = \frac{1}{2} g^{\alpha\beta} \partial_\alpha \varphi\, \partial_\beta \varphi - \frac{1}{2} m^2 \varphi^2$

$$\frac{\delta \sqrt{g}}{\delta g^{\alpha\beta}} \Lambda = -\frac{1}{2} \sqrt{g}\, g_{\alpha\beta}\, \Lambda$$

$$\frac{\sqrt{g}}{2} T_{\alpha\beta} = -\frac{1}{2} \left(\sqrt{g}\, g_{\alpha\beta} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi\, \partial_\nu \varphi - \frac{1}{2} m^2 \varphi^2 \right)\right) + \frac{1}{2} \sqrt{g}\, \partial_\alpha \varphi\, \partial_\beta \varphi$$

$$T_{\alpha\beta} = \partial_\alpha \varphi \, \partial_\beta \varphi + \frac{1}{2} g_{\alpha\beta} m^2 \varphi^2 - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi$$

flat space: $T_{00} = \partial_0 \phi \, \partial_0 \phi + \frac{1}{2} m^2 \varphi^2 - \frac{1}{2}\left(\partial_0 \varphi \, \partial_0 \varphi - (\vec{\nabla}\varphi)^2\right)$

$$= \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{1}{2} m^2 \varphi^2 \overset{\checkmark}{=} \text{WSB}$$

Maxwell field: (now use $\frac{\delta}{\delta g^{\mu\nu}}$)

$$\delta\left(\sqrt{g}\, g^{\alpha\gamma} g^{\beta\delta} F_{\alpha\beta} F_{\gamma\delta}\right)$$

$$= -\frac{1}{2} g_{\mu\nu} \sqrt{g}\, g^{\alpha\gamma} g^{\beta\delta} F_{\alpha\beta} F_{\gamma\delta}$$

$$+ 2\sqrt{g}\, g^{\beta\delta} F_{\mu\beta} F_{\nu\delta}$$

$$T_{\mu\nu} = \frac{2}{\sqrt{g}}\left(-\frac{1}{4}\right) \delta \text{ (above)}$$

$$= -F_{\mu\beta} F_{\nu\delta}\, g^{\beta\delta} + \frac{1}{4} g_{\mu\nu} g^{\alpha\gamma} g^{\beta\delta} F_{\alpha\beta} F_{\gamma\delta}$$

~~traceless~~ $g^{\mu\nu} T_{\mu\nu} = 0$ ✓

flat space: $T_{00} = E^2 + \frac{1}{4}\cancel{2}(B^2 - E^2)$

$$\overset{\checkmark}{=} \frac{1}{2}(E^2 + B^2)$$

exercise: check Poynting vector, Maxwell stresses!

ii) Conservation law

Preliminary: We expect the conservation of energy-momentum to be tied up with invariance under translations. So: let us translate!

$x'^\mu - x^\mu = \delta x^\mu = \varepsilon^\mu$ small, vanishing at ∞

$$g'^{\mu\nu}(x') = \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta} g^{\alpha\beta}(x)$$

$$\cong g^{\mu\nu}(x) + \partial_\alpha \varepsilon^\mu g^{\alpha\nu} + \partial_\beta \varepsilon^\nu g^{\mu\beta}$$

$$g'^{\mu\nu}(x') \cong g'^{\mu\nu}(x) + \partial_\alpha g^{\mu\nu} \varepsilon^\alpha$$

Thus $\delta g^{\mu\nu} = g'^{\mu\nu}(x) - g^{\mu\nu}(x) = \partial_\alpha \varepsilon^\mu g^{\alpha\nu} + \partial_\beta \varepsilon^\nu g^{\mu\beta} - \partial_\alpha g^{\mu\nu}\varepsilon^\alpha$

(x): not x'

Now notice

$$\delta g^{\mu\nu} = g^{\alpha\mu}\nabla_\alpha \varepsilon^\nu + g^{\alpha\nu}\nabla_\alpha \varepsilon^\mu \quad \text{(Killing eq}^{\underline{n}}\text{)}.$$

$$\Big[= g^{\alpha\mu}\partial_\alpha \varepsilon^\nu + g^{\alpha\mu}\overbrace{\tfrac{1}{2} g^{\nu\beta}\left(\partial_\alpha g_{\beta\rho} + \partial_\rho g_{\alpha\beta} - \partial_\beta g_{\alpha\rho}\right)\varepsilon^\rho}^{\Gamma^\nu_{\alpha\rho}\varepsilon^\rho}$$

cancel.

$$+ g^{\alpha\nu}\partial_\alpha \varepsilon^\mu + g^{\alpha\nu}\tfrac{1}{2} g^{\mu\beta}\left(\partial_\alpha g_{\beta\rho} + \partial_\rho g_{\alpha\beta} - \partial_\beta g_{\alpha\rho}\right)\varepsilon^\rho$$

$$= g^{\alpha\mu}\partial_\alpha \varepsilon^\nu + g^{\alpha\nu}\partial_\alpha \varepsilon^\mu + g^{\alpha\mu} g^{\beta\nu}\partial_\rho g_{\alpha\beta}\,\varepsilon^\rho$$

$$= g^{\alpha\mu}\partial_\alpha \varepsilon^\nu + g^{\alpha\nu}\partial_\alpha \varepsilon^\mu - \partial_\rho g^{\alpha\beta}\,\varepsilon^\rho$$

The last step follows from differentiating:

$g_{\alpha\mu} g^{\mu\beta} = \delta^\beta_\alpha$; so $\partial_\lambda g_{\alpha\mu}\, g^{\mu\beta} + g_{\alpha\mu}\partial_\lambda g^{\mu\beta} = 0$

and multiplying by $g^{\rho\alpha}$, so $\partial_\lambda g_{\alpha\mu}\, g^{\mu\beta} g^{\rho\alpha} = -\partial_\lambda g^{\rho\beta}$]

Also write this as

$$\delta g^{\mu\nu} = \varepsilon^{\nu;\mu} + \varepsilon^{\mu;\nu}$$

Now we have, from invariance of action & symmetry of $T_{\mu\nu}$ (& its definition!)

$$0 = \int \sqrt{g}\, T_{\mu\nu}\, \varepsilon^{\mu;\nu}$$

But also

$$0 = \int \partial^{\nu}\left(\sqrt{g}\, T_{\mu\nu}\, \varepsilon^{\mu}\right)$$

$$= \int \sqrt{g}\, \nabla^{\nu}\left(T_{\mu\nu}\, \varepsilon^{\mu}\right)$$

$$= \int \sqrt{g}\, \nabla^{\nu} T_{\mu\nu}\, \varepsilon^{\mu} + \underbrace{\int \sqrt{g}\, T_{\mu\nu}\, \varepsilon^{\mu;\nu}}_{=0}$$

Since this holds for any ε^{μ}, we conclude

$$\nabla^{\nu} T_{\mu\nu} = 0.$$