1.8 Newtonian Limit

We should be able to identify Newtonian gravity (and fix the coupling constant) by looking at the situation with nearly flat space and only $T_{00} = \rho$ significant. (For now of course we ignore the cosmological term.)

The stationary action condition gives

$$K\,(R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R) = \tfrac{1}{2} T_{\alpha\beta}$$

or $$2K\, R_{\alpha\beta} = T_{\alpha\beta} - \tfrac{1}{2} T\, g_{\alpha\beta}$$ [N.B. $g_{00} \cong c^2 \gg g_{ij}$]

Focus on R_{00}

$$2K\, R_{00} = \rho/2$$

In $R^{\cdot}{}_{\cdots}$, terms with $\Gamma\Gamma$ are higher order.

Also terms with $\frac{\partial}{\partial x^0} \sim \frac{1}{c}\frac{\partial}{\partial x^i}$ are small.

This is

$$R_{00} \approx \partial_\gamma \Gamma^\gamma_{00} - \partial_0 \Gamma^\gamma_{0\gamma}$$

$$\overset{\approx}{\Longrightarrow} \; \frac{1}{2} \vec{\nabla}^2 g_{00}$$

and finally

$$2K \vec{\nabla}^2 g_{00} = \rho$$

To interpret this, write $g_{00} = 1 + \varepsilon$ ($+ g^{00} = 1 - \varepsilon$).

The action density of matter is perturbed by

$$\frac{\delta \sqrt{g}\Lambda}{\delta g^{00}} \delta g^{00} \approx -\frac{\rho}{2} \varepsilon$$

This looks like the Newtonian coupling if

$\varepsilon = 2\phi$. The equation ~~of~~

$$2K \vec{\nabla}^2 g_{00} = \rho \Longrightarrow 4K \vec{\nabla}^2 \phi = \rho$$

This fixes $K = \frac{1}{16\pi G_N}$.

$$\left[\; \phi \approx -\frac{G_N M}{r} ; \; \nabla\phi = \frac{GM}{r^2} \underset{\text{Gauss}}{\Rightarrow} \int \nabla^2\phi \approx \underset{M/4K}{\overset{\shortparallel}{4\pi G M}} \; \right]$$

Note on the appendices and scholia:

These are not necessarily self-contained; specifically, they refer to facts about quantum field theory and the standard model that are not assumed elsewhere in the course. Don't worry if not everything is clear (or even meaningful) to you at this stage. Ask me if you're curious!

Central material:

1) the notion of local Lorentz invariance, ~~and mui~~ vierbein, and R ~~so~~ recipe (App. 1-2).

2) the idea that we are building up a model of the world, and are free to try anything (Scholium 4).

Appendix 1: Spinors and Local Lorentz Invariance

Spinor fields play a very important role in fundamental physics, so we ~~know~~ must learn how to ~~realize~~ treat them in general relativity.

The essential thing is to define γ^a-matrices. ~~They~~ transform under Lorentz transformations, $(SO(3,1))$ not $GL(4)$. Indeed the defining relation

$$\gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab}$$

refers to the flat-space Minkowski metric, and the transformation law

$$\gamma'^a = S^{-1}(\Lambda)\gamma^a S(\Lambda) = \Lambda^a{}_b \gamma^b$$

requires ~~an~~ S that does not exist for general

$\Lambda \in GL(4)$.

So we postulate local Lorentz invariance under transformations of this form, with $\Lambda(x)$ a function of x, $\Lambda(x) \in SO(3,1)$.

To connect this structure to the metric we introduce a vierbein

$$e^a_\mu(x)$$

such that

$$\eta_{ab}\, e^a_\mu(x)\, e^b_\nu(x) = g_{\mu\nu}(x) \qquad (\text{"square root" of metric})$$

$$g^{\mu\nu}\, e^a_\mu(x)\, e^b_\nu(x) = \eta^{ab} \qquad (\text{"moving frame"})$$

Now e.g. we can form the Dirac equation

$$(\gamma^a e^\mu_a D_\mu + m)\psi = 0 .$$

But D_μ needs discussion. We want invariance under local Lorentz transformations. This requires (exercise!)

$$D_\mu S(\Lambda(x)) = S(\Lambda(x)) D_\mu$$

– a typical "gauge invariance". We solve this problem "as usual" by introducing a gauge potential

$$\omega_\mu^{ab}(x) \in so(3,1)$$

$\uparrow$ Lie algebra ; $\omega_\mu^{ab}(x) = -\omega_\mu^{ba}(x)$

such that (in matrix notation)

$$\omega'_\mu(x) = \Lambda^{-1}\omega_\mu(x)\Lambda - \Lambda^{-1}\partial_\mu\Lambda$$

and writing

$$D_\mu = \nabla_\mu + \omega_\mu \cdot \tau$$

where the τ matrices provide the appropriate representation of the symmetry, e.g. $\sigma^{ab} = \frac{i}{4}[\gamma^a\gamma^b]$ for spin $\frac{1}{2}$, a the identity for spin 1.

To avoid introducing additional structure (cf. Scholium 4), we demand

$$D_\mu e^a_\nu = 0$$

$$\parallel$$

$$\partial_\mu e^a_\nu - \Gamma^\alpha_{\mu\nu} e^a_\alpha + \omega_\mu{}^a{}_b \, e^b_\nu \qquad (*)$$

leading to a unique determination

$$\omega_\mu{}^a{}_c = - e^\nu_c \, \partial_\mu e^a_c + e^\nu_c \, e^a_\alpha \, \Gamma^\alpha_{\mu\nu}$$

Appendix 2: Moving Frames Method + Recipe

The discussion of Appendix 1 is not as profound as it should be: $\omega_\mu{}^{ab}$ should be "more primitive" than

Γ! This is worth pursuing, since it leads to a beautiful analogy and useful formulae.

We can eliminate Γ from the defining relation (*) for ω by antisymmetrizing in $\mu \leftrightarrow \nu$. Thus

$$\partial_\mu e^a_\nu - \partial_\nu e^a_\mu = \omega_\mu^{ac} e_{c\nu} - \omega_\nu^{ac} e_{c\mu}$$

To solve for ω we go through a slight rigamarole, reminiscent of what we did to get ~~from~~ Γ from $\nabla g = 0$.

$$+\ e_{a\rho}(\partial_\mu e^a_\nu - \partial_\nu e^a_\mu) = \omega_\mu^{ac} e_{a\rho} e_{c\nu} - \omega_\nu^{ac} e_{a\rho} e_{c\mu}$$

$$-\ e_{a\mu}(\partial_\nu e^a_\rho - \partial_\rho e^a_\nu) = -\omega_\nu^{ac} e_{a\mu} e_{c\rho} + \omega_\rho^{ac} e_{a\mu} e_{c\nu}$$

$$+\ e_{a\nu}(\partial_\rho e^a_\mu - \partial_\mu e^a_\rho) = \omega_\rho^{ac} e_{a\nu} e_{a\mu} - \omega_\mu^{ac} e_{a\nu} e_{c\rho}$$

$$e_{a\rho}\partial_\mu e^a_\nu - e_{a\rho}\partial_\nu e^a_\mu - e_{a\mu}\partial_\nu e^a_\rho + e_{a\mu}\partial_\rho e^a_\nu + e_{a\nu}\partial_\rho e^a_\mu - e_{a\nu}\partial_\mu e^a_\rho = 2\omega_\mu^{ac} e_{a\rho} e_{c\nu}$$

using $\omega_\mu^{ac} = -\omega_\mu^{ca}$. Multiplying both sides by $e^{e\rho} e^{f\nu}$, we get

$$2\omega_\mu^{ef} = \overset{A}{e^{f\nu}\partial_\mu e^e_\nu} - \overset{B}{e^{f\nu}\partial_\nu e^e_\mu} - \overset{C}{e_{a\mu} e^{\rho e} e^{\nu f}\partial_\nu e^a_\rho}$$

$$+ \overset{C}{e_{a\mu} e^{e\rho} e^{f\nu}\partial_\rho e^a_\nu} + \overset{B}{e^{\rho e}\partial_\rho e^f_\mu} - \overset{A}{e^{\rho e}\partial_\mu e^f_\rho}$$

or

$$\boxed{\omega_\mu^{ef} = \frac{e^{f\nu}}{2}\left(\partial_\mu e^e_\nu - \partial_\nu e^e_\mu + e_{a\mu} e^{e\rho}\partial_\rho e^a_\nu - (e \leftrightarrow f)\right)}$$

Now we can construct a curvature by differentiating (say) a space-time scalar, local Lorentz vector field

$$(D_\mu D_\nu - D_\nu D_\mu)\phi^a \equiv R_{\mu\nu}{}^a{}_b\,\phi^b$$

This leads to

$$\boxed{\begin{aligned} \mathcal{R}_{\mu\nu}{}^{a}{}_{b} = {} & \partial_\mu \omega_\nu{}^{a}{}_{b} - \partial_\nu \omega_\mu{}^{a}{}_{b} \\ & + \omega_\mu{}^{a}{}_{c}\, \omega_\nu{}^{c}{}_{b} - \omega_\nu{}^{a}{}_{c}\, \omega_\mu{}^{c}{}_{b} \end{aligned}}$$

(but not too surprised)

Now you will be delighted to learn that this "gauge" curvature is intimately related to the Riemann curvature we had before; indeed

$$\boxed{R_{\mu\nu\alpha\beta} = \mathcal{R}_{\mu\nu}{}^{a}{}_{b}\, e_{a\alpha}\, e^{b}_{\beta}} \quad !$$

This actually gives the most powerful recipe for computing R.

The technique of introducing frames $e^a_\mu(x)$ to make the geometry "locally flat" was developed by E. Cartan, & it is called the moving frame method. It is usually presented in very obscure ways.

Scholium 1: Structure and Redundancy

Allowing a very general framework and demanding symmetry is an alternative to finding a canonical form that "solves" the symmetry. Thus, we consider general coordinate transformations but postulate a metric to avoid "solving" for local Lorentz frames and then pasting them together. ~~Th~~

Vierbeins or moving frames make this much explicit.

Fixing down to a *specific* frame is gauge fixing in the usual sense.

Scholium 2: Why General Relativity?

Ordinary spin-1 gauge fields are in danger of producing wrong-metric particles, or "ghosts". This is because covariant quantization conditions (commutation relations) for the different polarizations:

$$[a_\mu^\dagger, a_\nu] = -g_{\mu\nu}$$

if normal for the space-like pieces are abnormal for the time-like, and vice versa. Gauge symmetry allows one to show the wrong-metric excitations don't couple (e.g. one ~~could~~ can choose $A_0 = 0$ gauge).

Similary, general covariance / local Lorentz symmetry are required for consistent quantum theory of spin 2.

Scholium 3: General Relativity vs. Standard Model

In the SM symmetry + weak cutoff dependence (renormalizability) greatly restrict the possible couplings. One requires a linear manifold of fields (i.e., $\phi_1(x) + \phi_2(x)$ is allowed if $\phi_1(x)$ and $\phi_2(x)$ are).

In GR there is still symmetry, + minimal coupling ⇒ weak cutoff dependence below the Planck scale. However the field manifold is not linear ($g^1_{\mu\nu} + g^2_{\mu\nu}$ may not be invertible, hence not allowed) + there is a ~~[illegible]~~ dimensional fundamental coupling.

It looks like an effective theo, with spontaneous symmetry breaking, i.e. $e^a_\mu(x) = \langle\ ?\ \rangle$, like the σ-model.

Scholium 4 : Gauge and Dilaton Extensions

The standard assumption of Riemannian geometry is $\nabla_\alpha g^{\mu\nu} = 0$. However we might want to relax this, to incorporate additional symmetry. Some important physical ideas have arisen (or have natural interpretations) along this line.

Weyl wanted to unify electromagnetism with gravity. He postulated

$$\nabla_\alpha g_{\mu\nu} = S_\alpha g_{\mu\nu}$$

and the symmetry

$$g'_{\mu\nu}(x) = \lambda(x) g_{\mu\nu}(x)$$

$$S'_\alpha(x) = S_\alpha + \partial_\alpha \lambda$$

This was the historical origin of gauge invariance!

He wanted to identify f_α with the electromagnetic potential A_α. That has problems, but the ideas are profound.

Symmetry of the type $g'_{\mu\nu}(x) = \lambda(x)\, g_{\mu\nu}(x)$ arises in modern conformal field theory and string theory. It is called "Weyl symmetry". Given the importance of massless particles, and the idea that massive fundamental particles can acquire mass by spontaneous symmetry breaking, this idea of a scale symmetry retains considerable appeal.

A closely related variant is

$$\nabla_\alpha g_{\mu\nu} = \partial_\alpha \phi \; g_{\mu\nu}$$

$$g'_{\mu\nu} = \lambda g_{\mu\nu}$$

$$\phi' = \phi + \lambda$$

~~The~~ ϕ-fields of this sort are called "dilatons".