2.1 Model Spaces: Construction

Spaces (and spacetimes) of high symmetry play a very important role in cosmological model-building, and as examples ("solvable models") of GR. The most important ones can be considered as different odd sorts of spheres, so we start with that.

a) 3 d sphere

$$\sum_{i=1}^{4} x_i^2 = R^2$$

spherical coordinates

$$x_1 = R\cos \chi/R \qquad x_3 = R\sin \chi/R \sin\theta \cos\phi$$

$$x_2 = R\sin \chi/R \cos\theta \qquad x_4 = R\sin \chi/R \sin\theta \sin\phi$$

$$dl^2 = \sum dx_i^2 \underset{\text{ex.}}{=} d\chi^2 + R^2 \sin^2 \chi/R \, (d\theta^2 + \sin^2\theta \, d\phi^2)$$

["quasi-flat"] Write $x_4^2 = R^2 - r^2$

$$dx_4 = \frac{r\,dr}{x_4} \; ; \; dx_4^2 = \frac{r^2 dr^2}{R^2 - r^2}$$

$$d\ell^2 = \sum_{i=1}^{3} dx_i^2 + dx_4^2 = dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) + \frac{r^2 dr^2}{R^2 - r^2}$$

$$= \frac{dr^2}{1 - r^2/R^2} + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \qquad (*)$$

$$\text{or} \quad = R^2 \left(\frac{du^2}{1-u^2} + u^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \right), \quad u \equiv \frac{r}{R}.$$

[conformal] It is often useful to write

$$ds^2 = \cancel{f(x) ds^2_{flat}} \quad f^2(x) \; ds^2_{flat}$$

if that is possible. (Penrose diagrams ... later.)

Starting from our previous form, we will have this if we use η in place of r such that

$$\frac{dr^2}{1 - r^2/R^2} = f^2 \, d\eta^2$$
$$r^2 = f^2 \eta^2$$

$$\Longrightarrow \quad \frac{d\eta}{\eta} = \frac{dr}{r\sqrt{1 - r^2/R^2}}$$

leading to $\eta = \tan\frac{u}{2}$ with $\sin u = r/R$

[write $r = R\sin u$; $\dfrac{dr}{r\sqrt{1-r^2/R^2}} = \dfrac{du}{\sin u} = d\ln\tan\frac{u}{2}$

$$\parallel$$

$$\frac{d\eta}{\eta} = d\ln\eta \quad]$$

or – after some algebra –

$$d\ell^2 = \frac{4R^2}{(1+\eta^2)^2}\left(d\eta^2 + \eta^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right)$$

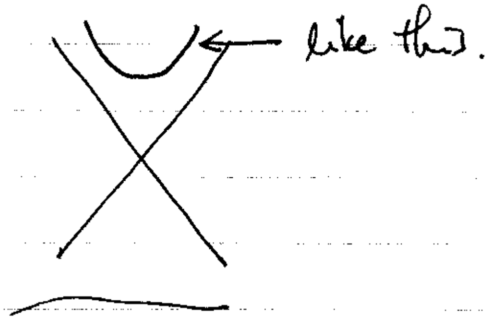
[$f^2 = \dfrac{r^2}{\eta^2}$; $\sin^2 u = r^2/R^2$

$$\parallel$$

$$4\sin^2\frac{u}{2}\cos^2\frac{u}{2} = 4\left(\frac{\eta^2}{1+\eta^2}\right)\left(\frac{1}{1+\eta^2}\right) \quad]$$

The sphere supports the symmetry $SO(4)$.

2) 3d hyperboloid (space of constant negative curvature)



Minkowski space-time

$$X_0^2 - \sum_{i=1}^{3} X_i^2 = R^2$$

'spherical'

$X_0 = R \cosh \chi/R$ etc.

$X_1 = R \sinh \chi/R \cos\theta$

$$dl^2 = -dX_0^2 + d\vec{X}^2 = d\chi^2 + R^2 \sinh^2\frac{\chi}{R}\,(d\theta^2 + \sin^2\theta\, d\phi^2)$$

quasi-flat

$|\vec{x}| = r$

$X_0^2 = R^2 + r^2$ ...

$$dl^2 = \frac{dr^2}{1 + r^2/R^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$= R^2\left(\frac{du^2}{1+u^2} + u^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right)$$ with $u \equiv r/R$

conformal

$$dl^2 = \frac{4}{(1-\eta^2)^2}\left(d\eta^2 + \eta^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right)$$

with $\eta = \tanh\frac{u}{2}$, $\sinh u = \frac{r}{R}$

~~Symmetry: SO(3,1)~~

Supports symmetry $SO(3,1)$, i.e.

"Lorentz" symmetry, acting purely spatially!

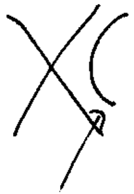
To bring this out, use

$$x_0 = \sqrt{r^2+R^2}\cosh\lambda \qquad x_2 = r\cos\phi$$
$$x_1 = \sqrt{r^2+R^2}\sinh\lambda \qquad x_3 = r\sin\phi$$

$$dl^2 = \frac{1}{1+r^2/R^2}\,dr^2 + r^2\,d\phi^2 + (r^2+R^2)\,d\lambda^2$$

"Translations" $\lambda \to \lambda + \text{const.}$, ~~leave~~ corresponding to boosts in the original variables, leave this invariant.

3) de Sitter spacetime



$$x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 = -R^2$$

like this

$$x_1 = R \cosh\frac{\chi}{R} \cos\lambda$$
$$x_2 = R \cosh\frac{\chi}{R} \sin\lambda \cos\theta$$
$$x_3 = R \cosh\frac{\chi}{R} \sin\lambda \sin\theta \cos\phi$$
$$x_4 = R \cosh\frac{\chi}{R} \sin\lambda \sin\theta \sin\phi$$
$$x_0 = R \sinh\frac{\chi}{R}$$

$$ds^2 = dx_0^2 - \Sigma dx_i^2$$

spherical

$$= d\chi^2 - R^2 \cosh^2\frac{\chi}{R} \underbrace{\left(d\lambda^2 + \sin^2\lambda\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right)}_{\text{unit 3-sphere}}$$

N.B. : exponential expansion!; minimum radius; spheres

quasi

$$x_0^2 = r^2 - R^2 \;; \qquad dx_0^2 = \frac{r^2 dr^2}{r^2 - R^2}$$

$$ds^2 = \frac{dr^2}{r^2/R^2 - 1} - r^2\left(d\lambda^2 + \sin^2\lambda\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right)$$

$\boxed{\text{light-front}}$ Separate out planes $(x_2, x_3, x_4) = \vec{x}_\perp$;

$$x_+ = x_0 + x_1 , \quad x_- = x_0 - x_1 .$$

$$\underbrace{x_0^2 - x_1^2}_{x_+ x_-} - \vec{x}_\perp^2 = -R^2 \quad ; \quad x_- = \frac{\vec{x}_\perp^2 - R^2}{x_+}$$

$$ds^2 = dx_+ \, dx_- - d\vec{x}_\perp^2 \; ;$$

$$dx_- = \frac{2\vec{x}\cdot d\vec{x}}{x_+} - \frac{dx_+}{x_+^2}(\vec{x}_\perp^2 - R^2)$$

So $$ds^2 = dx_+ \left(\frac{2\vec{x}\cdot d\vec{x}}{x_+} - \frac{dx_+}{x_+^2}(\vec{x}^2 - R^2) \right) - d\vec{x}^2$$

To remove the ugly cross-term, introduce

$\vec{v} = f(x_+)\vec{x}$. So $d\vec{v} = f' dx_+ \vec{x} + f \, d\vec{x}$,

$$d\vec{v}^2 = f'^2 \vec{x}^2 dx_+^2 + 2ff' \vec{x}\cdot d\vec{x} + f^2 d\vec{x}^2 ,$$

$$-d\vec{x}^2 = \frac{1}{f^2}\left(-d\vec{v}^2 + f'^2 \vec{x}^2 dx_+^2 + 2ff' \vec{x}\cdot d\vec{x}\right).$$

The x-term cancels if $\frac{f'}{f} = -\frac{1}{x_+}$, $f = \pm\frac{1}{x_+}$ $(x_+ > 0)$.

Thus with $\vec{v} \equiv \frac{\vec{x}}{x_+}$,

$$ds^2 = dx_+ \left(-\frac{dx_+}{x_+^2} (\vec{x}^2 - R^2) \right) - x_+^2 d\vec{v}^2 + \frac{1}{x_+^2} \vec{x}^2 dx_+^2$$

$$= R^2 \frac{dx_+^2}{x_+^2} - x_+^2 d\vec{v}^2$$

Now with $x_+ \equiv R e^{t/R}$

$$ds^2 = dt^2 - R^2 e^{t/R} d\vec{v}^2$$

- <u>expanding</u> ~~[illegible]~~ <u>flat</u> spatial metric!

$\boxed{\text{conformal}}$

$$ds^2 = R^2 x_+^2 \left(\frac{dx_+^2}{x_+^4} - d\vec{v}^2 \right)$$

so with $x_+ \equiv \frac{1}{u}$

$$ds^2 = R^2 \frac{1}{u^2} (du^2 - d\vec{v}^2)$$

de Sitter space has the symmetry $SO(4,1)$, from the ~~de~~ hyperboloid definition.

In the light-front coordinates we have translation symmetries $\vec{v} \to \vec{v} + \overrightarrow{\text{const}}$. Where do these sit? See Appendix 3.

We'll have much more to say about de Sitter later (inflation).

2.2 FRW (Friedman-Robertson-Walker) spacetimes

These are ~~obtained~~ constructed by choosing one of the maximally symmetric spaces ~~for~~ and letting ~~the~~ its overall scale vary with time.

Thus

$$ds^2 = dt^2 - a^2(t)\, dl^2$$

with $$dl^2 = \frac{du^2}{1+Ku^2} + u^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

where

$k=1$ hyperbolic sections
$k=0$ flat sections
$k=-1$ spherical sections

These model spacetimes are homogeneous and isotropic, but evolving. They supply interesting 1st models for the observed universe, averaged over large scales.

2.3 Curvature Calculations

Our master formulas (with correct signs)

are

$$\omega_\mu^{\ ef} = \frac{e^{f\nu}}{2}\left(\partial_\mu e_\nu^e - \partial_\nu e_\mu^e + e_{a\mu} e^{e\rho}\partial_\rho e_\nu^a\right)$$
$$-(e \leftrightarrow f)$$

$$R_{\mu\nu}^{\ \ \alpha\beta} = -F_{\mu\nu}^{\ \ ab}\, e_a^\alpha e_b^\beta$$

$$F_{\mu\nu}^{\ \ ab} = \partial_\mu \omega_\nu^{\ ab} - \partial_\nu \omega_\mu^{\ ab} - \omega_{\mu\ c}^{\ a}\,\omega_{\nu\ b}^{\ c} + \omega_{\nu\ c}^{\ a}\,\omega_{\mu\ b}^{\ c}$$

This is best exploited (for $g_{\mu\nu}$ diagonal) by using quasi-cartesian vierbeins

$$e_\alpha^a = \delta_\alpha^a g_a \quad (\text{so } e^{a\alpha} = \eta^{a\alpha} g_a^{-1}, \text{ etc.})$$

η^{ef}

1st term: $\dfrac{\eta^{f\nu}}{2} g_f^{-1}\delta_\nu^e \partial_\mu g_e \longrightarrow 0$ (symmetric in $e \leftrightarrow f$)

2nd term: $-\frac{1}{2}\eta^{f\nu} g_f^{-1}\delta_\mu^e \partial_\nu g_e = -\frac{1}{2}\eta^{f\nu} g_f^{-1}\delta_\mu^e \partial_\nu g_e$

3rd term: $\cancel{+\frac{1}{2}\eta^{f\nu} g_f^{-1}\eta_{a\mu} g_a g^e g_a^{-1}}$

3rd term: $\frac{1}{2}\underbrace{\eta^{f\nu}\eta_{a\mu}\eta^{e\rho}\eta^{a}_{\ \nu}}_{\Rightarrow f,\nu,\mu,a \text{ all } =}\ \cancel{g_f^{-1}}\cancel{g_a}\, g_e^{-1}\,\partial_\rho g_f$

$= \frac{1}{2}\delta^f_\mu\,\eta^{e\rho}\,g_e^{-1}\,\partial_\rho g_f$

= antisym. of 1st.

So

$$\omega_\mu^{\ ef} = \delta^f_\mu\,\eta^{e\rho}\,g_e^{-1}\,\partial_\rho g_f - \delta^e_\mu\,\eta^{f\rho}\,g_f^{-1}\,\partial_\rho g_e$$

mnemonic: "μ matches one index, the other differentiates its g"

Ex 1 2d sphere (warm-up)

$e_\theta^{\ 1} = 1 = g_1;\ e_\phi^{\ 2} = \sin\theta = g_2$

$\omega_\theta^{\ 12} = 0 \qquad (\delta^e_\mu \text{ only, but } \partial_2 g_1 = 0$

$\omega_\phi^{\ 12} = +\cos\theta \qquad (\delta_2^{f=2}\,\eta^{1\rho}\,g_1^{-1}\,\partial_1 g_2\ )$

$F_{\theta\phi}^{\ \ 12} = \partial_\theta\,\omega_\phi^{\ 12} + \text{vanishing}$

$= -\sin\theta$

$R_{\theta\phi}^{\ \ \theta\phi} = -\left(-\sin\theta\ \overset{1}{e_1^{\ \theta}}\ \overset{\frac{1}{\sin\theta}}{e_2^{\ \phi}}\right) = 1$

By the way, this is the gauge field of a magnetic monopole (gauge group $SO(2) \approx U(1)$!

<u>Ex. 2</u> 3d sphere

$$e^1_\chi = 1 \quad e^2_\theta = \sin\chi \quad e^3_\phi = \sin\chi \sin\phi$$

$$\underbrace{\begin{aligned} \omega_\chi^{12} &= 0 \\ \omega_\chi^{13} &= 0 \\ \omega_\chi^{23} &= 0 \end{aligned}}_{\text{by inspection}} \qquad \underbrace{\begin{aligned} \omega_\theta^{12} &= \cos\chi \\ \omega_\theta^{13} &= 0 \\ \omega_\theta^{23} &= 0 \end{aligned}}_{\substack{\text{as before} \\ \text{+ by inspection}}} \qquad \underbrace{\begin{aligned} \omega_\phi^{12} &= 0 \\ \omega_\phi^{13} &= \cos\chi \sin\theta \\ \omega_\phi^{23} &= \cos\theta \end{aligned}}_{\text{use mnemonic, --}}$$

$$F^{12}_{\chi\theta} = \partial_\chi \omega^{12}_\theta = -\sin\chi$$

$$F^{13}_{\chi\phi} = \partial_\chi \omega^{13}_\phi = -\sin\chi \sin\theta$$

$$F^{13}_{\theta\phi} = \partial_\theta \omega^{13}_\phi - \omega^{12}_\theta \omega^{23}_\phi = \cos\chi\cos\theta - \cos\chi\cos\theta = 0$$

$$F^{23}_{\theta\phi} = \partial_\theta \omega^{23}_\phi - \omega^{21}_\theta \omega^{13}_\phi = -\sin\theta + \cos^2\chi \sin\theta$$

$$= -\sin^2\chi \sin\theta$$

Thus

$$-F_{\mu\nu}{}^{ab} = e_\mu^a e_\nu^b - e_\mu^b e_\nu^a$$

(N.B. the antisymmetry on indices μ,ν + a,b is automatic!)

$$\text{or} \quad R_{\mu\nu}{}^{\alpha\beta} = \delta_\mu^\alpha \delta_\nu^\beta - \delta_\mu^\beta \delta_\nu^\alpha$$

$$\text{and} \quad R_\nu{}^\beta = 2\,\delta_\nu^\beta$$

$$\text{and} \quad R = 6$$

Ex. 3 FRW cosmology (~~flat case~~ spatially flat case)

$$e_t^0 = 1 \qquad e_i^c = \delta_i^c \, a(t)$$ ↙ expansion factor

$$ds^2 = dt^2 - a(t)^2 \, d\vec{x}^{\,2}$$

Note: mid-Latin are spatial, early-Latin are internal

The only non-zero ω is

$$\omega_i{}^{0c} = \delta_i^c \, \dot{a} \qquad !$$

The non-vanishing components of the field strength are

$$F_{0i}{}^{0c} = \partial_0 \omega_i{}^{0c} = \delta_i^c \, \ddot{a}$$

$$F_{ij}{}^{cd} = -\omega_i{}^{c0} \omega_j{}^{0d} + \omega_j{}^{c0} \omega_i{}^{0d}$$

$$= (\delta_i^c \delta_j^d - \delta_j^c \delta_i^d) \, \dot{a}^2$$

leading to the Ricci tensor components

$$R_0{}^0 = -3 \frac{\ddot{a}}{a} \qquad (= -F_{0i}{}^{0c} e_c^i)$$

$$R_i{}^l = -F_{ij}{}^{cd} e_c^l e_d^j - F_{0i}{}^{0c} e_c^l$$

$$= \left(-2 \frac{\dot{a}^2}{a^2} - \frac{\ddot{a}}{a}\right) \delta_i^l$$

$$R = -6 \frac{\dot{a}^2}{a^2} - 6 \frac{\ddot{a}}{a}$$

2.3 FRW Dynamics

The field equations (for $g^{\alpha\beta}$) are

$$R^{\mu}_{\ \nu} - \frac{1}{2}\delta^{\mu}_{\ \nu} R = 8\pi G\, T^{\mu}_{\ \nu}$$

We interpret $T^{0}_{\ 0} = \rho$, $T^{i}_{\ j} = -p\,\delta^{i}_{j}$

(Check 1: for electromagnetism, $T^{\mu}_{\ \mu} = 0$ and $p = \frac{1}{3}\rho$.)

From the preceding calculation, then

$$\text{(A)} \qquad 8\pi G\rho = 3\frac{\dot a^2}{a^2}$$

$$\left[\, 8\pi G(\rho+p) = 2\left(\frac{\dot a^2}{a^2} - \frac{\ddot a}{a}\right)\right.$$

$$8\pi G\, p = -2\frac{\ddot a}{a} - \frac{\dot a^2}{a^2}$$

Another important + appealing equation comes from differentiating the first of these and eliminating:

$$8\pi G\dot\rho = 6\frac{\dot a}{a}\left(\frac{\ddot a}{a} - \frac{\dot a^2}{a^2}\right)$$

$$= -3\; 8\pi G\frac{\dot a}{a}(\rho+p)$$

or simply

(B) $$\dot\rho = -3(\rho + p)\frac{\dot a}{a}$$

Another interesting thing is to see who's responsible for acceleration:

$$8\pi G\,(\rho + 3p) = -6\,\frac{\ddot a}{a}$$

There is a simple interpretation of (A) & (B).

(A): Imagine a ~~small~~ test particle along

for the ride. Gravity "outside" cancels (Birkhoff theorem). Conservation of particle's energy:

$$\frac{m}{2}\overbrace{r^2\dot a^2}^{v^2} - \frac{G\,\frac{4\pi}{3}\rho\, r^3 a^3\; m}{r} = m k r^2$$

$$\dot{a}^2 - \frac{8\pi G\rho}{3} a^2 = k$$

We have this with $k=0$: neutral binding, critical "escape velocity"!

The non-zero values of k arise in FRW spaces with hyperbolic $(k>0)$ and spherical $(k<0)$ spatial sections – see pb. set!

(B):  Imagine work done by expanding fluid against pressure; take it from mass-energy:

$$\left(\frac{4\pi}{3}\rho a^3 r^3\right)^{\cdot} = -p \left(\frac{4\pi}{3} a^3 r^3\right)^{\cdot}$$

$$\Downarrow$$

$$(\rho a^3)^{\cdot} = -p(a^3)^{\cdot}$$

$$\Downarrow$$

$$\dot{\rho} = -3(\rho+p)\frac{\dot{a}}{a}$$

Appendix 3: Translations within $SO(4,1)$

Write metric in block form

$$-g = \left(\begin{array}{c|c} J & 0 \\ \hline 0 & 1 \end{array}\right) \qquad J \equiv \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad 1 \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ etc.}$$

The condition for a near-identity transformation

$S = 1 + \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right)$ to leave the metric invariant

is

$$S^T g S = g \;; \quad \left(\begin{array}{c|c} a^T & c^T \\ \hline b^T & d^T \end{array}\right)\left(\begin{array}{c|c} J & 0 \\ \hline 0 & 1 \end{array}\right) + \left(\begin{array}{c|c} J & 0 \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right) \approx 0$$

or (to 1^{st} order)

$$\begin{aligned} a^T J + J a &= 0 \\ J b - c^T &= 0 \\ b^T J - c &= 0 \\ d^T + d &= 0. \end{aligned}$$

With $a = d = 0$ the transformations

$1 + \left(\begin{array}{c|c} 0 & b \\ \hline b^T J & 0 \end{array}\right)$ translate vectors $\begin{pmatrix} r \\ s \end{pmatrix}$ by

$\begin{pmatrix} bs \\ b^T J r \end{pmatrix}$, i.e. with things spelled out completely:

$$b = \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \eta & \phi \end{pmatrix}; \text{ so } \quad b^T J = \begin{pmatrix} -\alpha & \delta \\ -\beta & \eta \\ -\gamma & \phi \end{pmatrix}$$

and

$$\Delta r = b s = \begin{pmatrix} \alpha s_1 + \beta s_2 + \gamma s_3 \\ \delta s_1 + \eta s_2 + \phi s_3 \end{pmatrix}$$

$$\Delta s = b^T J r = \begin{pmatrix} -\alpha r_1 + \delta r_2 \\ -\beta r_1 + \eta r_2 \\ -\gamma r_1 + \phi r_2 \end{pmatrix}$$

Transformations with $\delta = -\alpha$, $\eta = -\beta$, $\phi = -\gamma$ leave $r_1 + r_2$ fixed while translating s through

$$\Delta s = -\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} (r_1 + r_2) \quad !$$

So

$$\frac{\vec{s}}{r_1 + r_2}$$

is translated in the conventional way. In our previous notation this is

$$\frac{\vec{s}}{r_1 + r_2} = \frac{(X_2, X_3, X_4)}{X_0 + X_1} = \frac{\vec{X}_\perp}{X_+} = \vec{v} \quad ! \quad \left(\text{which explains } \vec{v}\right)$$