2.5 Canonical Examples

One can solve the system

$$\frac{8\pi G}{3}\rho = \left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2} \quad (\equiv H^2)$$

Hubble parameter

$$\dot\rho = -3(\rho + p)\frac{\dot a}{a}$$

given an equation of state $p = p(\rho)$ and initial values for ρ, a.

A) Flat "dust" (matter dominated)

$\Rightarrow k=0, p=0$.

$$\dot\rho = -3\rho\frac{\dot a}{a} \Rightarrow \rho a^3 = \text{const.} \equiv \frac{3}{4\pi} M_0 a_0^3$$

interpretation: mass inside a reference volume conserved

Put it back: $$\frac{2M_0 a_0^3}{a^3} = \frac{\dot a^2}{a^2}$$

$$\dot a^2 a = \text{const.} \Rightarrow a^{1/2} da \sim dt, \; a = ct^{2/3}$$

Notes i) Expansion from (curvature) singularity

ii) relation of expansion rate with density

$$\frac{8\pi G}{3}\rho_c = \left(\frac{\dot{a}}{a}\right)^2 = H^2$$

~~[illegible]~~

This actually tests $k=0$, <u>not</u> the eq^{n} of state.

B) Flat "radiation"

$\Rightarrow k=0 \quad p = \frac{1}{3}\rho \quad$ (traceless $T^{\mu}_{\ \mu}$)

$$\dot{\rho} = -4\rho\frac{\dot{a}}{a} \Rightarrow \rho a^4 = \text{const.}$$

interpretation: entropy (or <u>action</u>) is conserved

photon wavelength stretched $\propto a$

$\Rightarrow$ energy $\propto \frac{1}{a}$: <u>not</u> conserved

$$\dot{a}^2 a^2 = \text{const.} \Rightarrow a = c\,t^{1/2}$$

c) de Sitter : flat, vacuum-dominated

$T^{\mu\nu} \propto g^{\mu\nu}$

"Ether" that respects Lorentz invariance, e.g. static scalar field energy

$$T^{\mu}_{\ \nu} = \lambda \delta^{\mu}_{\ \nu} \Rightarrow \quad \rho = \lambda \ , \ p = -\lambda$$

$$\dot{\rho} = 0 \quad (!)$$

(negative pressure ~~does~~ positive work, supplies rest-energy to fill new volume!)

$$\frac{8\pi G}{3} \lambda = \left(\frac{\dot{a}}{a}\right)^2 \Rightarrow a \propto e^{Ht} \ ; \ H = \sqrt{\frac{8\pi G}{3} \lambda}$$

Note: i) This FRW realization gives the light-front version of de Sitter, as above.

ii) The $k<0$ ~~gives~~ version would lead us to the sphere version we discussed earlier; $k>0$ leads to a hyperboloid version (exercise).

D) Einstein static universe

$\dot{a}=0$ using $\rho+p=0$ and spatial curvature

$\Rightarrow \rho=\lambda,\ a=a_0$ with

$$\frac{8\pi G}{3}\lambda = -\frac{k}{a_0^2}$$

This can be solved with $k<0$: spherical spatial universe.

<u>Notes</u>: i) Tremendously important historically:

a) Universe "closed, but without boundary".

b) Appearance of λ.

c) Assumption of spatial homogeneity + isotropy

BUT

ii) "Island Universe" = galaxy X

iii) static X

iv) Unstable ! - wants to break out into deSitter

This was what Einstein called his "greatest blunder". It really was a blunder, given point iv).

Scholium 5 Simplified Matter

One cannot couple in an arbitrary source to a given space; the condition $\nabla_\alpha T^\alpha{}_\beta = 0$ is necessary for consistency.

In our derivation of the FRW framework this is guaranteed, ~~of cou~~ since we solved the field equations, ~~for~~ which incorporate the Bianchi identity.