2.6 Global Structure; Causal Relations

Some of the most interesting, novel issues introduced by dynamic space-time relate to global structure, especially causal structure - initial conditions, who can affect who. To analyze this it is convenient to exploit the conformal invariance of the light-ray condition $ds = 0$, to map onto simpler easier-to-visualize realizations. This is the method of Penrose diagrams. It is easiest to appreciate in the context of concrete examples.

(Flat) Minkowski Space-Time

This defines "normal" causal structure.

With $x_\pm = t \pm r$,

$$ds^2 = dx_+ dx_- - \frac{(x_+ - x_-)^2}{4} \underbrace{(d\theta^2 + \sin^2\theta \, d\phi^2)}_{\parallel \atop d\Omega_2^2}$$

Write $x_+ = 2\tan\frac{u}{2}$, $x_- = 2\tan\frac{v}{2}$.

We want to consider a slice at constant $\theta, \phi \equiv \theta_0, \phi_0$ but also include planar continuation through $r = 0$ to $\theta, \phi = \pi - \theta_0, \phi_0 + \pi$ with negative r. Indeed, these angular coordinates reverse $\vec{x} = r(\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi)$.

With this understanding, $-\pi < u < \pi$, $-\pi < v < \pi$, covers space-time.

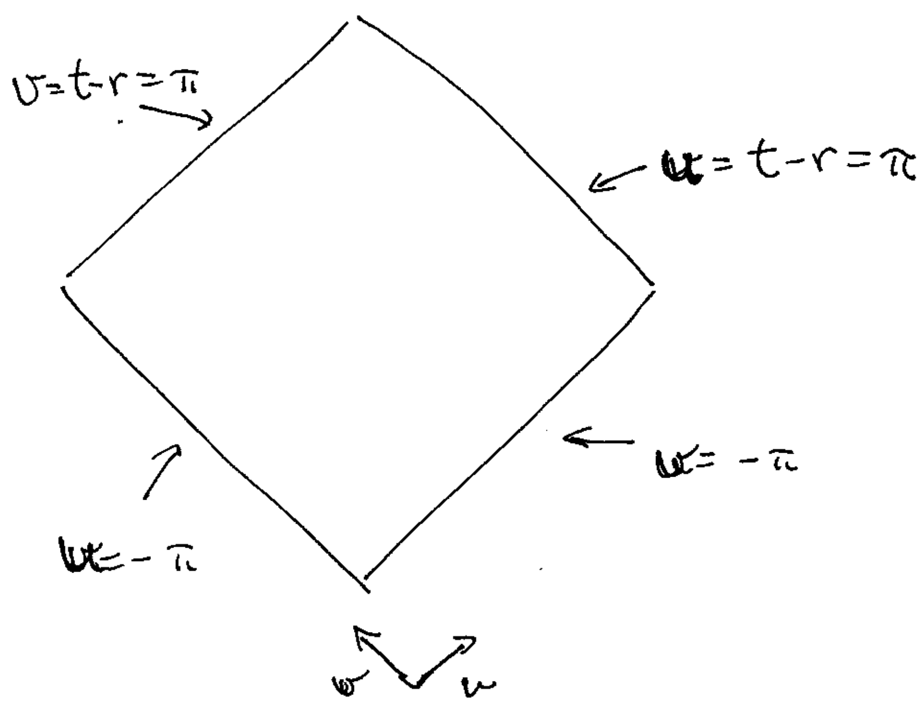
Since $dx_+ = \frac{1}{\cos^2\frac{u}{2}}\, du$, $dx_- = \frac{1}{\cos^2\frac{v}{2}}\, dv$

and $\left(\tan\frac{u}{2} - \tan\frac{v}{2}\right)^2 = \frac{\sin\left(\frac{u-v}{2}\right)^2}{\cos^2\frac{u}{2}\cos^2\frac{v}{2}}$ we find

$$ds^2 = \frac{1}{\cos^2\frac{u}{2}\cos^2\frac{v}{2}}\left\{du\,dv - \sin^2\frac{u-v}{2}\,d\Omega^2\right\}$$

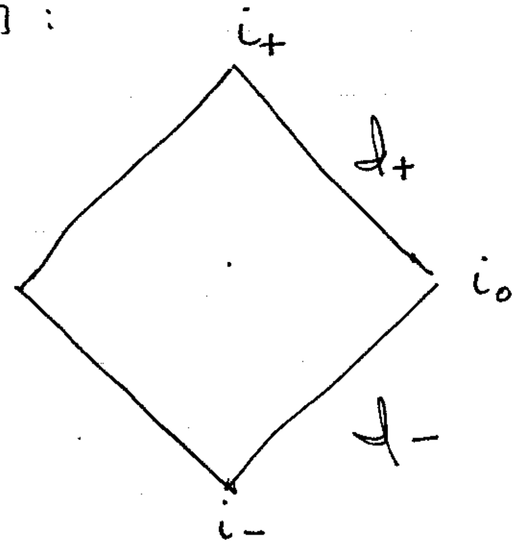
$$\equiv \frac{1}{\cos^2\frac{u}{2}\cos^2\frac{v}{2}}\, d\bar{s}^2$$

Now re-introduce t, r



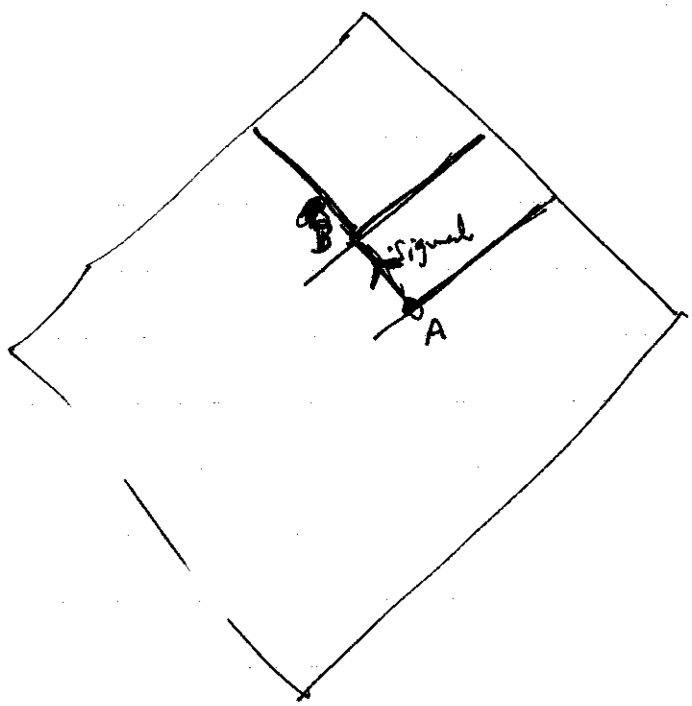
— a diamond.

Concepts:



i_+: Future timelike infinity. This is where time-like ~~paths~~ geodesics — corresponding to the world-lines of massive particles — asymptote in the future.

$\mathscr{I}_+$: Future null infinity. Pronounced "Scri+" This is where null geodesics (light rays) converge. Note that it is <u>not</u> a single point: outgoing light-beams can't exchange signals!



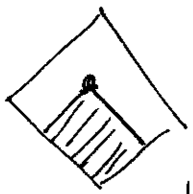
(If A signals B, B cannot signal back to A).

i_0 : spatial infinity

Similarly i_-, $\mathscr{I}_-$

Note:

i) Causal past of observer is backward light-cone. It always contains i_- $\mathscr{I}$,



though not all of $\mathscr{I}_-$!

ii) As an observer goes to i_+, his or her backward light-cone comes to encompass the whole space-time.

iii) Any spacelike surface extending between to two i_0 "points" is a Cauchy surface. That is, it ~~contains~~

~~the a fault continuation~~ of intersects all time-like or nulls paths leading to P ~~in their pas~~ at a time prior to P. Thus it contains the information necessary to predict events at P.

"Normal" FRW Models

$ds^2 = dt^2 - a^2 d\vec{x}^2 \quad a = t^p \quad p < 1.$

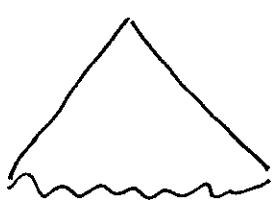
Introduce $d\eta = \frac{dt}{a}$ $\quad \eta = \int_0^t \frac{dt}{a}$

convergent!

So $ds^2 = a^2 (d\eta^2 - d\vec{x}^2)$.

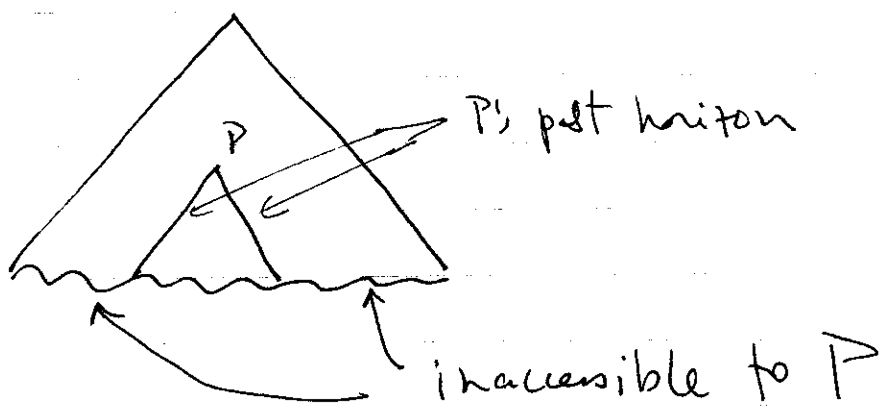
~~But~~ singularity of R at $t = 0$.

Now proceed as before. The result is



$u + v = 0$

It is no longer true that all (timelike) ~~[illegible]~~ observers "see" each other at i_-: indeed, there is no i_-! New parts of the singularity are always coming into view!



Elementary perspective: given t_0, maximum visible $\vec{x}$:

$$dt = \pm a \, dx$$

$$dx = \int_0^{t_0} \frac{dt}{a} < \infty !$$

de Sitter

$$ds^2 = d\eta^2 - \cosh^2\eta \,(d\chi^2 + \sin^2\chi \, d\Omega^2)$$

To conformalize, define

$$d\sigma = \frac{d\eta}{\cosh\eta} = \frac{2e^\eta d\eta}{e^{2\eta}+1} = \frac{2d\tau}{\tau^2+1}$$

with $\tau = e^\eta$

$$= 2\,d\tan^{-1}\tau = 2\,d(\tan^{-1}e^\eta).$$

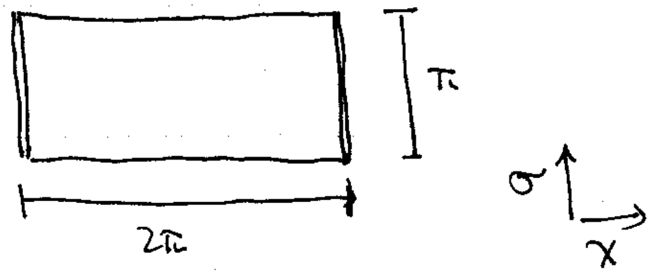
so $\sigma = 2\tan^{-1}e^\eta$

As $\eta \to -\infty$ $\sigma \to 0$,
$\eta \to +\infty$ $\sigma \to \pi$.

The 'polar' coordinate χ satisfies $0 \le \chi \le \pi$.

~~The~~ $\chi = 0$ is a coordinate singularity, like $r=0$ in Minkowski space-time. Again, we extend through to negative χ by $\theta \to \pi - \theta$, $\phi \to \phi + \pi$.

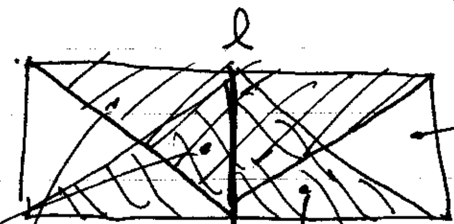
Finally we arrive at



— The two sides ‖ are to be identified (exercise), so it's a cylinder.

There are no singularities, but there are horizons + interesting causal structure.

Consider comoving ~~station~~ worldline l



— never-never land for l.

region that can affect l

region that l can affect

normal give-and-take possible
— but critical periods!

It gets lovely in the far future: