2.7 Measures of Space-Time Geometry

~~###~~ Many practical measures of space-time geometry are based on its effect on light propagation. In analyzing this, it is extremely helpful to exploit the conformal symmetry of the Maxwell Lagrangian.

$$\int \sqrt{g}\, g^{\alpha\gamma} g^{\beta\delta} \underbrace{F_{\alpha\beta} F_{\gamma\delta}}_{\text{no } g_{..} \text{ here}}$$

invariant under
$g_{\mu\nu} \to f(x)\, g_{\mu\nu}$

All the FRW model universes are conformally equivalent to flat Minkowski space. Here for simplicity we'll assume the spatial sections are flat, so

$$ds^2 = dt^2 - a(t)^2\, d\vec{x}^2$$

The conformal flatness is easy:

$$ds^2 = a^2 \left(d\eta^2 - d\vec{x}^{\,2} \right)$$

with $d\eta \equiv \frac{dt}{a}$

Thus the plane-wave solutions $A_\mu \epsilon\, e^{i(\vec{k}\vec{x} - \omega\eta)}$ (with $\omega = |\vec{k}|$) are valid in FRW, ~~etc~~ as are other familiar properties of radiation, so long as we use ~~η~~ in place of t.

i) A first consequence is the <u>red</u> <u>shift</u> relation.

$$\text{perceived frequency} = \frac{\cancel{d}\,\text{phase}}{dt} = \omega \frac{d\eta}{dt} = \frac{\omega}{a}$$

Thus

$$\frac{\text{frequency as observed } \cancel{\text{frequency at emission}}\ (t_2)}{\text{frequency at emission } (t_1)} = \frac{a(t_1)}{a(t_2)} \equiv \frac{1}{1+z}$$

The conventional red-shift parameter z is

0 now and positive in the past, for an expanding universe. It can be measured by matching observed spectra to know (laboratory) lines.

~~iii) luminosity distance~~ Note that for small z,
$$\frac{\delta\omega}{\omega} = -z.$$

Now we should get distance measures, so we can get the expansion factor as a function of distance or (equivalently) look-back time. There are 3 classic ways to do this

ii) luminosity distance

Suppose we manage to identify a class of standard candles - objects with the same intrinsic power output (luminosity) L wherever and whenever they occur. (It is believed that certain types of supernovae approximate this.)

Then the observed luminosity depends on geometric and redshift factors as follows.

$$\mathcal{L}_{obs.} \propto \frac{L_0}{r_1^2 a(t_2)^2 (1+z)^2} \equiv \frac{L_0}{\ell_L^2}$$

where

$r_1 \equiv$ comoving coordinate of source

$\ell_L \equiv$ luminosity distance

$L_0 \equiv$ intrinsic luminosity

For small distances the observed power flux ~~area~~ goes as $1/(\text{distance})^2$, which ~~as~~ — $z \to 0$, $r_1 a \to$ distance —

I am assuming the observer is at the origin.

which motivates this definition

To make this useful we need to eliminate r_1. The light rays propagate ~~from~~ along lines $\eta \pm r = \text{const.}$, so the ray connecting (η_1, r_1) to $(\eta_2, 0)$ has

"now"

$r_1 = \eta_2 - \eta_1$.

The factors $\frac{1}{(1+z)^2}$ arise from

$\frac{1}{1+z}$: redshift

$\frac{1}{1+z}$: time dilation : the energy transmitted in one cycle of phase is spread out over a longer time, since the phase develops $\propto \eta$.

Thus

$$\boxed{l_L = a(t_2)\,(\eta_2 - \eta_1)\,(1+z)}$$

We can expand for small z, leading to

$$(*) \quad H l_{\cancel{p}} \approx z + z^2\left(\tfrac{1}{2} - \tfrac{1}{2} q_0\right)$$ [algebra: next page]

with $H \equiv \dot{a}/a$, $q_0 \equiv -\dfrac{\ddot{a}\,a}{\dot{a}^2}$

$\uparrow$ deceleration parameter

(evaluated at t_2, i.e. now).

For a given source, assuming it (represents) a standard candle, we can observe z and $l_{\cancel{p}}$, so several such sources permit one to determine H and q_0 observationally.

$(*)$ is called the Hubble relation.

Algebra for Hubble relation:

From

$$a(1) = a(2) + \dot{a}(2)(t_1 - t_2) + \frac{1}{2}\ddot{a}(2)(t_1-t_2)^2$$

we have

$$\frac{1}{1+z} = \frac{a(1)}{a(2)} = 1 - H\,\Delta t - \frac{1}{2}H^2 q_0 (\Delta t)^2$$

$t_2 - t_1$

$$z - z^2 = H\Delta t + \frac{1}{2}H^2 q_0 (\Delta t)^2$$

So (solving iteratively)

$$H\,\Delta t \approx z - z^2\left(1 + \frac{1}{2}q_0\right).$$

$$\eta_2 - \eta_1 = \int_{t_1}^{t_2} \frac{dt}{a(2) + \dot{a}(2)(t - t_2) + \frac{1}{2}\ddot{a}(2)(t-t_2)^2}$$

$$= \frac{1}{a(2)}\int_{t_1}^{t_2}\frac{dt}{1 - H(t_2 - t) - \frac{1}{2}H^2 q_0 (t_2-t)^2}$$

$$\approx \frac{1}{a(2)}\int_{t_1}^{t_2} dt\,\left(1 + H(t_2 - t) - \frac{1}{2}H^2 q_0 (t_2 - t)^2 + H^2(t_2-t)^2\right)$$

$$= \frac{1}{a(2)}\left[\Delta t + \frac{1}{2}H(\Delta t)^2 + \cdots\right] \approx \frac{1}{a(2)}\left[\frac{z}{H} - \frac{z^2}{H}\left(1+\frac{1}{2}q_0\right) + \frac{1}{2}\frac{z^2}{H}\right]$$

and $\ell_2 H = a(2)(\eta_2 - \eta_1)(1+z) = z + z^2\left(\frac{1}{2} - \frac{1}{2}q_0\right)$

For a matter dominated FRW phase

$$a \propto t^{2/3} \Rightarrow q_0 = -\frac{\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)}{\left(\frac{2}{3}\right)^2} = \frac{1}{2}$$

For a λ-dominated phase

$$a \propto e^{Ht} \Rightarrow q_0 = -1$$

Thus curvature in an l_L vs. z plot can distinguish these cases. Recent observations indicate a significant λ component.

iii) angle distance

$$\delta_{obs} = \frac{D}{a(1) r_1} \equiv \frac{D}{l_A}$$

observed angle

intrinsic size

fractional size

So $l_A = a(1) r_1 = (1+z)^2 l_L$

iv) Source Counts

For this we need persistent candles

$$dN \propto \text{physical volume} \propto a(1)^3 r_1^2 dr_1 \, d\Omega$$

$$\sim \frac{1}{(1+z)^3} (r_2 - r_1)^2 \, dr_1 \quad d\Omega$$

$$\underset{\text{previous algebra}}{\sim} \quad \frac{1}{(1+z)^3} \frac{1}{H^2} \left[z - z^2 \left(\frac{1}{2} + \frac{1}{2} q_0 \right) \right]^2 \times dz \left[1 - (1+q_0) z \right]$$

...

v) Surface brightness

Brightness of standard candle

$$\propto \frac{\text{luminosity}}{(\text{angular size})^2} \propto \frac{1}{(1+z)^4}$$

since $l_A = (1+z)^2 l_L$

This can be used to check "standard candle" hypothesis.

2.8 Olbers paradox

In static homogeneous Euclidean space a sprinkling of persistent standard candles gives ∞ total luminosity:

$$\int^R \frac{L_0}{r^2} r^2 dr \propto R$$

We have instead

$$\int L\, dN \sim \int \frac{L_0}{(1+z)^2 (\eta_2-\eta_1)^2} \frac{1}{(1+z)^3} (\eta_2-\eta_1)^2 \, d\eta$$

$$\sim \int \frac{d\eta}{(1+z)^5}$$

This is highly convergent as $z \to \infty$, which (for normal ~~~~ Big Bangs) happens at finite η. Why? - finite age of universe, shrinking horizon, red shift + time dilation.

(smaller visible volume)

3.1 Effect of ~~[illegible]~~ expansion on matter

"master equation" (Bose-Einstein condensate
Klein-Gordon field
⟶ "Schrodinger eqn."

$$\int \sqrt{g}\left(\tfrac{1}{2} g^{\mu\nu}\partial_\mu \varphi\, \partial_\nu \varphi - \tfrac{1}{2} m^2 \varphi^2\right)$$

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi} = \partial_\mu\left(\sqrt{g}\, g^{\mu\nu}\partial_\nu \varphi\right)$$

$$\frac{\delta \mathcal{L}}{\delta \varphi} = - m^2 \varphi$$

$$\frac{1}{\sqrt{g}}\,\partial_\mu\left(\sqrt{g}\, g^{\mu\nu}\partial_\nu \varphi\right) = - m^2 \varphi$$

(generally: RHS $\rightarrow \frac{\delta V}{\delta \varphi}$)

$$ds^2 = \underset{\substack{\uparrow \\ \text{local time}}}{dt^2} - a^2(t)\, d\vec{x}^{\,2}$$

$$\sqrt{g} = a^3 \quad g^{00} = 1 \quad g^{ij} = -\frac{1}{a^2}\delta^{ij}$$

$$\ddot{\varphi} + 3\frac{\dot{a}}{a}\dot{\varphi} - \frac{1}{a^2}(\vec{\nabla}\varphi)^2 = - m^2 \varphi$$

$(\partial^2\varphi/\partial t^2)$ ⟵ cosmic viscosity!

Examples, simple cases

i) $\left(\frac{\vec{\nabla}\varphi}{a^2}\right)^2$ negligible

$$\varphi = A(t)$$

$$\ddot{A} + 3\frac{\dot{a}}{a}\dot{A} = -m^2 A$$

damped harmonic oscillator

a) $m \gg \dot{a}/a$

1st approx: $A = \alpha e^{-imt}$

$\alpha = $ const.

2nd approx:

$$2\dot{\alpha}(-im) + 3\frac{\dot{a}}{a}(-im\alpha) = 0$$

$$\frac{\dot{\alpha}}{\alpha} = -\frac{3}{2}\frac{\dot{a}}{a}$$

$$\alpha = \alpha_0 / a^{3/2}$$

T^0_0 energy density $\simeq \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}m^2\varphi^2$

$\simeq \alpha^2 m^2 \propto m^2/a^3$

Interpretation:

$$\frac{(\nabla\varphi)^2}{a^2} \ll m^2 \iff \vec{p}^{\,2} \ll m^2$$

$\Downarrow$

non-relativistic matter ("dust")

So $\rho \sim \frac{1}{a^3}$ is WSB.

b) $\frac{\dot a}{a} \gg m$

$$\ddot A + 3\frac{\dot a}{a}\dot A \approx 0$$

$$(\dot A a^3)^{\cdot} = 0$$

$$\dot A a^3 = \text{const.}$$

$$\dot A \xrightarrow[a \to \infty]{} 0.$$ "frozen" or "stuck"

next approx.

$$3\frac{\dot{a}}{a}\,\dot{\alpha} = -m^2\alpha$$

$$\dot{\alpha}/\alpha = -3m^2/H$$

"

$$(\ln\alpha)^{\cdot}$$

$$\frac{\alpha(t_2)}{\alpha(t_1)} = e^{-3m^2\int_{t_1}^{t_2}\frac{1}{H(\tau)}d\tau}$$

φ

N.B. : in de Sitter space, dies eventually

ii) $\vec{\nabla}\varphi$ back in, $\dot{a}/a$ small

1st ~~approx~~:

$$\cancel{\ddot{\varphi} - \frac{\vec{\nabla}\varphi^2}{a^2} = -m^2\varphi^2}$$

$$\varphi = A(t)\, e^{i[\hat{n}\cdot\vec{x}\,k - \int^t \omega(\tau)d\tau]}$$

k constant $\begin{cases}\text{symmetry ~~[illegible]~~ allows it} \\ \text{makes tractable analysis}\end{cases}$

$$-i\dot{\omega} - \omega^2 + \frac{k^2}{a^2} - 2\frac{i\dot{A}}{A}\omega - 3\frac{\dot{a}}{a}i\omega = -m^2$$

~~1st approx:~~

a) m negligible

1st approximation : $\cancel{\omega = m}$ $\omega = \frac{k}{a}$

$\Longrightarrow$ "redshift" again

2nd approximation (imaginary part)

$$2\frac{\dot{A}}{A} = -3\frac{\dot{a}}{a} - \frac{\dot{\omega}}{\omega} = -2\frac{\dot{a}}{a}$$

$$A \propto \frac{1}{a}$$

$$\rho \propto A^2 \left(\omega^2 \text{ or } \frac{k^2}{a^2}\right) \propto \frac{1}{a^4}$$

This is WSB for radiation.

Note: for this non-conformal scalar there are corrections, unlike for Maxwell eq$^{\underline{n}}$.

b) m dominant

1st approx : $\omega = m$

2nd approx: $2\frac{\dot{A}}{A} = -3\frac{\dot{a}}{a}$

as before

iii) $\vec{\nabla}\varphi$ back in, fast expansion

m dominant – done previously

k dominant

$$3\frac{\dot{a}}{a}\dot{A} + \frac{k^2}{a^2}A = 0$$

$$\dot{A}/A = -3\,{k^2/a^2}/H$$

$$\frac{A(t_2)}{A(t_1)} = e^{-\int_{t_1}^{t_2}\frac{k^2}{a^2H}}$$

Similar to m case: for $k^2/a^2 \ll H$ it is almost stuck, but dies eventually, for de Sitter.

3.2 Effect of Expansion on Free Thermal Distributions

For a massless particle $E = \sqrt{m^2+p^2} = p = \frac{k}{a}$.

Thus a Boltzmann factor e^{-E_1/T_1} matches e^{-E_2/T_2} with $T_2 = \frac{a(1)}{a(2)} T_1$

$$\left[k/a_1 T_1 = k/a_2 T_2 \right]$$

The measure $d^3x' \, d^3p' \sim a^3 \frac{1}{a^3} d^3x_0 \, d^3k$ also matches, in comoving coordinates.

Thus a thermal distribution evolves into a thermal distribution, with $T \propto 1/a$.

for massive particles

$$E = \sqrt{m^2 + p^2} \approx m + \frac{p^2}{2m} \rightarrow m + \frac{k^2}{2ma^2}$$

$\Rightarrow$ 1) The thermal distribution of kinetic energies $e^{-p^2/2mT}$ cools as $T_{kin} \propto \frac{1}{a^2}$.

2) The number distribution function $e^{-m/T}$ does not change (in comoving volume).

$T_{\#}$ (or μ, chemical potential) $\rightarrow$ const.

$\Rightarrow$ Cold relics.