Due: 30 Sept.

Problem Set I: Geodesics and Locally Geodesic (Riemann) Coordinates

1. Consider the expression for length of arc (or proper time) between two points:

$$\int_A^B ds = \int_A^B \left(\epsilon\, g_{\rho\sigma}\, dx^\rho dx^\sigma\right)^{1/2} = \int_A^B \left(\epsilon\, g_{\rho\sigma} \frac{dx^\rho}{du}\frac{dx^\sigma}{du}\right)^{1/2} du$$

with $\epsilon = \pm 1$ to make the $\sqrt{\ }$ real and positive, and u a parameter along the curve.

a) To get the equation for extremizing this length, regard it as a dynamical system with u the time. Show that the equation of motion can be written

$$\frac{d^2x^\mu}{du^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{du}\frac{dx^\sigma}{du} = \frac{1}{2L}\frac{dL}{du}\frac{dx^\mu}{du}$$

with $L \equiv \left(\epsilon\, g_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du}\right)^{1/2}$. [For now, do not worry about the possibility $L = 0$.]

b) ~~Ex~~ Show that this can be interpreted in the following way: if I ~~write~~ change the tangent vector along the curve by the covariant derivative, it stays parallel to itself. [i.e., interpret $\nabla_{\frac{d}{du}} = \frac{\partial x^\mu}{du} \nabla_\mu \frac{dx^\rho}{du} \propto \frac{dx^\rho}{du}$]

c) By choosing u as the solution of $du = (\epsilon g_{\mu\nu} dx^\mu dx^\nu)^{1/2}$, i.e. ~~the~~ $u = s$ the arc-length, derive the nice form

$$\frac{d^2 x^\mu}{ds} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds} = 0 \qquad (*)$$

d) Now considering (*) on its own, with $s \to \lambda$ show that $g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = \text{const.}$ is implied. This works even for const. $= 0$, which defines <u>null</u> geodesics. They solve

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0$$

$$g_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0$$

and define "light rays"

e) In part d), λ is an affine parameter. Show that $\lambda' = a\lambda + b$, with constant a, b, is still a good affine parameter. What happens to the equations if one chooses instead an arbitrary parameter?

Comment: These "geometric" concepts are, at this point, not related to physics. ~~Remembering Remem~~ The field Lagrangian contains the physics. ~~T~~ ~~Th~~ They emerge in the "geometric optics" — i.e., ~~light~~ high-frequency, short wavelengths, classical —

description of <u>solutions</u> of the wave equation. That will appear in the next problem set.

2. At a given point ~~$\mathcal{O}$~~ $\mathcal{O}$, pick a set of non-null basis vectors V^0, V^1, V^2, V^3 for the tangent space. Move by distances x^0 ~~along~~ V^0, x^1 along V^1 (parallel transported), etc. to define the point with parameters (x^0, x^1, x^2, x^3). Show that for small x^μ this is a ~~good~~ well-defined coordinate system and that in it $\Gamma^\mu_{\nu\rho} = 0$ at $\mathcal{O}$, which also implies $\frac{\partial g_{\alpha\beta}}{\partial x^\delta} = 0$ at $\mathcal{O}$.