

8.952 LECTURE 24

Looking Back: Conformal Newtonian Gauge

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Conformal Newtonian Gauge

The metric:

$$ds^2 = -(1 + 2\Phi)dt^2 + a^2(t) \left[(1 - 2\Psi)\delta_{ij} + \frac{\partial C_i}{\partial x^j} + \frac{\partial C_j}{\partial x^i} + D_{ij} \right] dx^i dx^j ,$$

where

$$\frac{\partial C_i}{\partial x^i} = 0 , \quad \frac{\partial D_{ij}}{\partial x^i} = 0 , \quad D_{ii} = 0 .$$

Leads to a Poisson-like equation, Weinberg's (5.3.26):

$$\frac{1}{a^2} \nabla^2 \Psi = 4\pi G \delta\rho - 12\pi G H (\bar{\rho} + \bar{p}) \delta u .$$

Question: What is the significance of the 2nd term on the right?

Note that δu is not even local! $\delta u_i = \frac{\partial \delta u}{\partial x^i} + \delta u_i^V$, where $\frac{\partial \delta u_i^V}{\partial x^i} = 0$.

Newtonian Gauge: Einstein Equations

R_{00} equation:

$$\frac{1}{a^2} \nabla^2 \Phi + \frac{6\ddot{a}}{a} \Phi + \frac{3\dot{a}}{a} (\dot{\Phi} + 2\dot{\Psi}) + 3\ddot{\Psi} = 4\pi G (\delta\rho + 3\delta p + \delta\pi_{ii}) ,$$

where the correction to the perfect fluid energy-momentum tensor is written

$$\delta\pi_{ij} \equiv \partial_i \partial_j \pi^S + \partial_i \pi_j^V + \partial_j \pi_i^V + \pi_{ij}^T ,$$

with

$$\frac{\partial \pi_i^V}{\partial x^i} = 0 , \quad \frac{\partial \pi_{ij}^T}{\partial x^i} = 0 , \quad \pi_{ii}^T = 0 , \quad \text{so } \delta\pi_{ii} = \nabla^2 \pi^S .$$

Note that the LHS of the R_{00} equation is not R_{00} . Perturbations from the metric in

$$T_{\mu\nu}^{\text{perfect}} = pg_{\mu\nu} + (\rho + p)u_\mu u_\nu$$

are brought to the LHS of the equation.

R_{0i} equation:

$$2 \left(\partial_i \dot{\Psi} + \frac{\dot{a}}{a} \partial_i \Phi \right) + \frac{1}{2} \nabla^2 \dot{C}_i = 2 \left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right) \delta u_i .$$

R_{ij} equation:

$$\begin{aligned} & \left[2 \left(\frac{\ddot{a}}{a} + 2 \frac{\dot{a}^2}{a^2} \right) \Phi + \frac{\dot{a}}{a} (\dot{\Phi} + 6\dot{\Psi}) + \ddot{\Psi} - \frac{1}{a^2} \nabla^2 \Psi \right] \delta_{ij} + \frac{1}{a^2} \partial_i \partial_j (\Phi - \Psi) \\ & - \frac{1}{2} (\partial_i \dot{C}_j + \partial_j \dot{C}_i) - \frac{3\dot{a}}{2a} (\partial_i \dot{C}_j + \partial_j \dot{C}_i) \\ & - \frac{1}{2} \left(\ddot{D}_{ij} + 3 \frac{\dot{a}}{a} \dot{D}_{ij} - \frac{1}{a^2} \nabla^2 D_{ij} \right) = 4\pi G (\delta p - \delta\rho + \delta\pi_{ii}) \delta_{ij} - 8\pi G \delta\pi_{ij} . \end{aligned}$$

Trace equation (R_{ii}):

$$\begin{aligned} & 3 \left[2 \left(\frac{\ddot{a}}{a} + 2 \frac{\dot{a}^2}{a^2} \right) \Phi + \frac{\dot{a}}{a} (\dot{\Phi} + 6\dot{\Psi}) + \ddot{\Psi} - \frac{1}{a^2} \nabla^2 \Psi \right] + \frac{1}{a^2} \nabla^2 (\Phi - \Psi) \\ & = 4\pi G (3\delta p - 3\delta\rho + \delta\pi_{ii}) . \end{aligned}$$

R_{00} equation:

$$\frac{1}{a^2} \nabla^2 \Phi + \frac{6\ddot{a}}{a} \Phi + \frac{3\dot{a}}{a} (\dot{\Phi} + 2\dot{\Psi}) + 3\ddot{\Psi} = 4\pi G (\delta\rho + 3\delta p + \delta\pi_{ii}) .$$

Rewrite of R_{ii} equation:

$$\begin{aligned} 6 \left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} \right) \Phi + 3\frac{\dot{a}}{a} (\dot{\Phi} + 6\dot{\Psi}) + 3\ddot{\Psi} + \frac{1}{a^2} \nabla^2 (\Phi - 4\Psi) \\ = 4\pi G (3\delta p - 3\delta\rho + \delta\pi_{ii}) . \end{aligned}$$

Subtracting R_{ii} equation from R_{00} equation and dividing by 4:

$$\frac{1}{a^2} \nabla^2 \Psi - 3H(\dot{\Psi} + H\Phi) = 4\pi G \delta\rho ,$$

where $H = \dot{a}/a$.

The extra term is related to Weinberg's expression through the R_{0i} equation:

$$2 \left(\partial_i \dot{\Psi} + \frac{\dot{a}}{a} \partial_i \Phi \right) + \frac{1}{2} \nabla^2 \dot{C}_i = 2 \left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right) \delta u_i \quad \xrightarrow{\text{extract scalar part}}$$

$$\partial_i (\dot{\Psi} + H\Phi) = \dot{H} \partial_i \delta u \quad \implies$$

$$(\dot{\Psi} + H\Phi) = \dot{H} \delta u = -4\pi G (\bar{\rho} + \bar{p}) \delta u ,$$

so

$$\frac{1}{a^2} \nabla^2 \Psi = 4\pi G \delta\rho - 12\pi G H (\bar{\rho} + \bar{p}) \delta u .$$

But why is this extra term here!?

Look at the divergence of the fluid velocity:

$$\begin{aligned} u^\mu{}_{;\mu} &= \frac{\partial u^\mu}{\partial x^\mu} + \Gamma^\mu_{\mu\lambda} u^\lambda \\ &= 3\bar{H} + \frac{1}{a^2} \partial_i u_i + \frac{3}{2} \bar{H} h_{00} - \frac{1}{a^2} \bar{H} h_{ii} + \frac{1}{2a^2} \dot{h}_{ii} - \frac{1}{a^2} \partial_i h_{i0} . \end{aligned}$$

In Newtonian gauge,

$$u^\mu{}_{;\mu} = 3\bar{H} + \frac{1}{a^2} \partial_i u_i - 3(\dot{\Psi} + H\Phi) .$$

You may recall that on 4/20/09 we used this relation to show that the perturbation variable

$$\mathcal{R}_q \equiv -\Psi_q + H \delta u_q$$

can be related to the variable

$$K \equiv a^2 \left[\frac{8\pi}{3} G \rho_{\text{loc}} - H_{\text{loc}}^2 \right] ,$$

where ρ_{loc} is the local energy density $\bar{\rho} + \delta\rho$, and $H_{\text{loc}} = \frac{1}{3} u^\mu{}_{;\mu}$ is the local expansion rate for the comoving fluid. Note that K is a local version of the Robertson-Walker curvature constant k , so it is immediately apparent that K is gauge-invariant, since any scalar which is constant in the background solution is gauge-invariant. It is also apparent that it is conserved in the long wavelength limit, since it is constant in a homogeneous solution. K was found to be related to \mathcal{R} by

$$K = -\frac{2}{3} \nabla^2 \mathcal{R} .$$

For points at rest in the coordinate system,

$$w^\mu{}_{;\mu} = 3\bar{H} - 3(\dot{\Psi} + H\Phi) = 3(\bar{H} + \delta H_{\text{coord}}) ,$$

where $\delta H_{\text{coord}} = -(\dot{\Psi} + H\Phi)$ is the perturbation in the local Hubble expansion rate for points at rest in the coordinate system. Since the local Hubble rate is perturbed, so is the local critical density:

$$\rho_{\text{cr}} = \frac{3H^2}{8\pi G} \implies \delta\rho_{\text{cr}} = \frac{3H\delta H}{4\pi G} = -\frac{3H}{4\pi G}(\dot{\Psi} + H\Phi) .$$

So the Poisson equation can be written

$$\begin{aligned} \frac{1}{a^2}\nabla^2\Psi &= 4\pi G\delta\rho + 3H(\dot{\Psi} + H\Phi) \\ &= 4\pi G(\delta\rho - \delta\rho_{\text{cr}}) . \end{aligned}$$

Thus the equation is telling us that the Newtonian potential responds to perturbations in the mass density relative to the local critical density.