8.952 LECTURE 24

Looking Back: Conformal Newtonian Gauge

May 4, 2009

Conformal Newtonian Gauge

The metric:

$$ds^2 = -(1+2\Phi)\mathrm{d}t^2 + a^2(t)\left[(1-2\Psi)\delta_{ij} + \frac{\partial C_i}{\partial x^j} + \frac{\partial C_j}{\partial x^i} + D_{ij}\right] \mathrm{d}x^i\,\mathrm{d}x^j \ ,$$

where

$$\frac{\partial C_i}{\partial x^i} = 0 \ , \quad \frac{\partial D_{ij}}{\partial x^i} = 0 \ , \quad D_{ii} = 0 \ .$$

Leads to a Poisson-like equation, Weinberg's (5.3.26):

$$\frac{1}{a^2}\nabla^2\Psi = 4\pi G\,\delta\rho - 12\pi G H(\bar{\rho}+\bar{p})\,\delta u \ .$$

Question: What is the significance of the 2nd term on the right?

Note that δu is not even local! $\delta u_i = \dfrac{\partial\,\delta u}{\partial x^i} + \delta u_i^V$, where $\dfrac{\partial\,\delta u_i^V}{\partial x^i} = 0.$

Newtonian Gauge: Einstein Equations

R_{00} equation:

$$\frac{1}{a^2}\nabla^2\Phi + \frac{6\ddot{a}}{a}\Phi + \frac{3\dot{a}}{a}(\dot{\Phi} + 2\dot{\Psi}) + 3\ddot{\Psi} = 4\pi G(\delta\rho + 3\delta p + \delta\pi_{ii}) \ ,$$

where the correction to the perfect fluid energy-momentum tensor is written

$$\delta\pi_{ij} \equiv \partial_i\partial_j\pi^S + \partial_i\pi^V_j + \partial_j\pi^V_i + \pi^T_{ij} \ ,$$

with

$$\frac{\partial\pi^V_i}{\partial x^i} = 0 \ , \quad \frac{\partial\pi^T_{ij}}{\partial x^i} = 0 \ , \quad \pi^T_{ii} = 0 \ , \quad \text{so} \quad \delta\pi_{ii} = \nabla^2\pi^S \ .$$

Note that the LHS of the R_{00} equation is not R_{00}. Perturbations from the metric in

$$T^{\text{perfect}}_{\mu\nu} = pg_{\mu\nu} + (\rho + p)u_\mu u_\nu$$

are brought to the LHS of the equation.

R_{0i} equation:

$$2\left(\partial_i\dot{\Psi}+\frac{\dot{a}}{a}\partial_i\Phi\right)+\frac{1}{2}\nabla^2\dot{C}_i=2\left(\frac{\ddot{a}}{a}-\frac{\dot{a}^2}{a^2}\right)\,\delta u_i\ .$$

R_{ij} equation:

$$\left[2\left(\frac{\ddot{a}}{a}+2\frac{\dot{a}^2}{a^2}\right)\Phi+\frac{\dot{a}}{a}\left(\dot{\Phi}+6\dot{\Psi}\right)+\ddot{\Psi}-\frac{1}{a^2}\nabla^2\Psi\right]\delta_{ij}+\frac{1}{a^2}\partial_i\partial_j(\Phi-\Psi)$$
$$-\frac{1}{2}\left(\partial_i\ddot{C}_j+\partial_j\ddot{C}_i\right)-\frac{3}{2}\frac{\dot{a}}{a}\left(\partial_i\dot{C}_j+\partial_j\dot{C}_i\right)$$
$$-\frac{1}{2}\left(\ddot{D}_{ij}+3\frac{\dot{a}}{a}\dot{D}_{ij}-\frac{1}{a^2}\nabla^2 D_{ij}\right)=4\pi G(\delta p-\delta\rho+\delta\pi_{ii})\,\delta_{ij}-8\pi G\delta\pi_{ij}\ .$$

Trace equation (R_{ii}):

$$3\left[2\left(\frac{\ddot{a}}{a}+2\frac{\dot{a}^2}{a^2}\right)\Phi+\frac{\dot{a}}{a}\left(\dot{\Phi}+6\dot{\Psi}\right)+\ddot{\Psi}-\frac{1}{a^2}\nabla^2\Psi\right]+\frac{1}{a^2}\nabla^2(\Phi-\Psi)$$
$$=4\pi G(3\,\delta p-3\delta\rho+\delta\pi_{ii})\ .$$

R_{00} equation:

$$\frac{1}{a^2}\nabla^2\Phi + \frac{6\ddot{a}}{a}\Phi + \frac{3\dot{a}}{a}(\dot{\Phi} + 2\dot{\Psi}) + 3\ddot{\Psi} = 4\pi G(\delta\rho + 3\delta p + \delta\pi_{ii}) \ .$$

Rewrite of R_{ii} equation:

$$6\left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2}\right)\Phi + 3\frac{\dot{a}}{a}\left(\dot{\Phi} + 6\dot{\Psi}\right) + 3\ddot{\Psi} + \frac{1}{a^2}\nabla^2(\Phi - 4\Psi)$$
$$= 4\pi G(3\,\delta p - 3\delta\rho + \delta\pi_{ii}) \ .$$

Subtracting R_{ii} equation from R_{00} equation and dividing by 4:

$$\frac{1}{a^2}\nabla^2\Psi - 3H(\dot{\Psi} + H\Phi) = 4\pi G\,\delta\rho \ ,$$

where $H = \dot{a}/a$.

The extra term is related to Weinberg's expression through the R_{0i} equation:

$$2\left(\partial_i\dot{\Psi} + \frac{\dot{a}}{a}\partial_i\Phi\right) + \frac{1}{2}\nabla^2\dot{C}_i = 2\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right)\,\delta u_i \quad \underset{\text{extract scalar part}}{\Longrightarrow}$$

$$\partial_i(\dot{\Psi} + H\Phi) = \dot{H}\partial_i\,\delta u \quad \Longrightarrow$$

$$(\dot{\Psi} + H\Phi) = \dot{H}\,\delta u \;= -4\pi G(\bar{\rho} + \bar{p})\,\delta u \;,$$

so

$$\frac{1}{a^2}\nabla^2\Psi = 4\pi G\,\delta\rho - 12\pi G H(\bar{\rho} + \bar{p})\,\delta u \;.$$

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$$\partial_i(\dot{\Psi}+H\Phi)=\dot{H}\partial_i\,\delta u \quad \Longrightarrow$$

$$(\dot{\Psi}+H\Phi)=\dot{H}\,\delta u \;=-4\pi G(\bar{\rho}+\bar{p})\,\delta u\ ,$$

so

$$\frac{1}{a^2}\nabla^2\Psi=4\pi G\,\delta\rho-12\pi GH(\bar{\rho}+\bar{p})\,\delta u\ .$$

But why is this extra term here!?

Look at the divergence of the fluid velocity:

$$u^{\mu}{}_{;\mu} = \frac{\partial u^{\mu}}{\partial x^{\mu}} + \Gamma^{\mu}_{\mu\lambda} u^{\lambda}$$

$$= 3\bar{H} + \frac{1}{a^2}\partial_i u_i + \frac{3}{2}\bar{H} h_{00} - \frac{1}{a^2}\bar{H} h_{ii} + \frac{1}{2a^2}\dot{h}_{ii} - \frac{1}{a^2}\partial_i h_{i0} \ .$$

In Newtonian gauge,

$$u^{\mu}{}_{;\mu} = 3\bar{H} + \frac{1}{a^2}\partial_i u_i - 3(\dot{\Psi} + H\Phi) \ .$$

You may recall that on 4/20/09 we used this relation to show that the perturbation variable

$$\mathcal{R}_q \equiv -\Psi_q + H\,\delta u_q$$

can be related to the variable

$$K \equiv a^2 \left[\frac{8\pi}{3} G \rho_{\rm loc} - H^2_{\rm loc} \right] ,$$

where $\rho_{\rm loc}$ is the local energy density $\bar{\rho} + \delta\rho$, and $H_{\rm loc} = \frac{1}{3} u^\mu{}_{;\mu}$ is the local expansion rate for the comoving fluid. Note that K is a local version of the Robertson-Walker curvature constant k, so it is immediately apparent that K is gauge-invariant, since any scalar which is constant in the background solution is gauge-invariant. It is also apparent that it is conserved in the long wavelength limit, since it is constant in a homogeneous solution. K was found to be related to $\mathcal{R}$ by

$$K = -\frac{2}{3} \nabla^2 \mathcal{R} .$$

For points at rest in the coordinate system,

$$u^{\mu}{}_{;\mu} = 3\bar{H} - 3(\dot{\Psi} + H\Phi) = 3(\bar{H} + \delta H_{\rm coord}) \ ,$$

where $\delta H_{\rm coord} = -(\dot{\Psi} + H\Phi)$ is the perturbation in the local Hubble expansion rate for points at rest in the coordinate system. Since the local Hubble rate is perturbed, so is the local critical density:

$$\rho_{\rm cr} = \frac{3H^2}{8\pi G} \quad \Longrightarrow \quad \delta\rho_{\rm cr} = \frac{3H\,\delta H}{4\pi G} = -\frac{3H}{4\pi G}(\dot{\Psi} + H\Phi) \ .$$

So the Poisson equation can be written

$$\frac{1}{a^2}\nabla^2\Psi = 4\pi G\,\delta\rho + 3H(\dot{\Psi} + H\Phi)$$

$$= 4\pi G(\delta\rho - \delta\rho_{\rm cr}) \ .$$

Thus the equation is telling us that the Newtonian potential responds to perturbations in the mass density relative to the local critical density.