MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.952: The Early Universe — March 22, 2009
Prof. Alan Guth

PROBLEM SET 4

DUE DATE: Friday, April 3, 2009, at 5 pm.

ANNOUNCEMENT: A second makeup class of 8.952 (in addition to the one held on March 20) will be held on Friday April 10, from 11:05 – 11:55 am, again in Room 1-273.

PROBLEM 1: CANONICAL FORMULATION OF GEODESIC MOTION IN GENERAL RELATIVITY *(15 points)*

Suppose that a timelike path is described by $x^\mu(s)$, where s is an arbitrary parameter that varies between s_1 and s_2. Then the proper time for this path can be written as

$$\tau = \int_{s_1}^{s_2} \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}}\,\mathrm{d}s\ , \tag{1}$$

where the metric is assumed to have the signature $(-+++)$.

(a) Show that the extremization of the proper time implies that

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1}{\sqrt{A}}g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\right] = \frac{1}{2\sqrt{A}}\frac{\partial g_{\lambda\sigma}}{\partial x^\mu}\frac{\mathrm{d}x^\lambda}{\mathrm{d}s}\frac{\mathrm{d}x^\sigma}{\mathrm{d}s}\ , \tag{2}$$

where

$$A = g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\ . \tag{3}$$

(b) Show that this formula can be used to obtain the more standard equation for geodesic motion,

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} = -\Gamma^\mu_{\lambda\sigma}\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}\ , \tag{4}$$

where

$$\Gamma^\mu_{\lambda\sigma} = \frac{1}{2}g^{\mu\rho}\left(\partial_\lambda g_{\rho\sigma} + \partial_\sigma g_{\rho\lambda} - \partial_\rho g_{\lambda\sigma}\right)\ . \tag{5}$$

[Suggestion: Since s is an arbitrary parameter, you can begin by choosing the special case $s = \tau$, where τ is the proper time, and hence $A = 1$. Then expand the left-hand side of Eq. (2) and rearrange.]

(c) If we take s to be t, the coordinate time variable, then Eq. (1) takes the form of Hamilton's principle of classical mechanics, where τ is interpreted as the action, with a Lagrangian

$$L\left(x^i, \frac{\mathrm{d}x^i}{\mathrm{d}t}\right) = -m\sqrt{-g_{\mu\nu}(x^i,t)\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}}\ , \tag{6}$$

where m is the mass of the particle. Note that $x^0 \equiv t$, so $\mathrm{d}x^0/\mathrm{d}t \equiv 1$. The factor of $-m$ on the right-hand side of Eq. (6) has no effect on the equations of motion, but is inserted so that the canonical momenta and Hamiltonian have familiar forms. Using this Lagrangian, show that the momenta conjugate to the position coordinates x^i, where i ranges from 1 to 3, are the covariant four-momentum components p_i, where

$$p_\mu \equiv g_{\mu\nu} p^\nu \equiv m g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \ . \tag{7}$$

(d) Show that the Hamiltonian, defined as usual by

$$H(x^i, p_i, t) = p_i \frac{\mathrm{d}x^i}{\mathrm{d}t} - L \ , \tag{8}$$

is given simply by

$$H = -p_0 \ , \tag{9}$$

where p_0 is the zeroth (i.e. time) component of the covariant four-momentum p_μ. Here the independent variables are $x^1, x^2, x^3, p_1, p_2, p_3$, and time t, with p_0 determined from the independent variables by requiring that

$$p^2 = g^{\mu\nu}(x^i, t) p_\mu p_\nu = -m^2 \ . \tag{10}$$

(e) Finally, show directly that Hamilton's equations,

$$\dot{x}^i = \frac{\partial H}{\partial p_i} \ , \qquad \dot{p}_i = -\frac{\partial H}{\partial x^i} \ , \tag{11}$$

are equivalent to the geodesic equation (2), with

$$\dot{x}^i = \frac{p^i}{p^0} \ . \tag{12}$$

Note that this is a standard canonical system, which therefore automatically evolves with a conserved phase space volume, as described by Liouville's theorem. Note also that the phase space volume is simply $\mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3 \mathrm{d}p_1 \mathrm{d}p_2 \mathrm{d}p_3$, as usual, with no corrections associated with the metric on the spacetime.

PROBLEM 2: LORENTZ-INVARIANCE OF THE PHASE SPACE VOLUME IN SPECIAL RELATIVITY *(10 points)*

The formalism developed in Problem 1 holds for any metric, which means it applies to the Minkowski metric, as a special case. In this problem we will restrict ourselves to this special case, and we will also assume for simplicity that the particles under discussion have nonzero mass. The nonzero mass implies that there exists a rest frame, which makes the analysis simpler.

Suppose that a phase space density $\mathcal{N}$ is defined so that the expected number of particles in a phase space volume $\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}p_1\mathrm{d}p_2\mathrm{d}p_3$ is given by

$$\text{Number of particles} = \mathcal{N}\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3\mathrm{d}p_1\mathrm{d}p_2\mathrm{d}p_3 \ . \tag{13}$$

Show that the phase space density is a Lorentz scalar, in the sense that in a moving (primed) frame, the phase space density $\mathcal{N}'(x'^i, p'_i, t')$ is related to that in the original frame by

$$\mathcal{N}'(x'^i, p'_i, t') = \mathcal{N}(x^i, p_i, t) \ , \tag{14}$$

where x'^i, p'_i, and t' are the values in the primed frame that correspond to x^i, p_i, and t in the original frame.

(Suggestion: One way to proceed is to first consider the special case where $\vec{p}$ is at rest, $p_i = 0$ for all i. Then each interparticle spacing behaves as a ruler that is boosted from its rest frame, so it contracts by a factor of γ in the direction of the motion. Show that the momentum-space volume changes in the right way to compensate, leaving the product $\mathrm{d}^3x'\,\mathrm{d}^3p'$ invariant. Once your have shown this result for the special case, explain how it implies the Lorentz-invariance of the phase space volume in the general case.)

PROBLEM 3: GENERAL COORDINATE INVARIANCE OF THE PHASE SPACE VOLUME IN GENERAL RELATIVITY *(20 points)*

In this problem we will show that the invariance of the phase space density holds not only for Lorentz transformations, but for arbitrary coordinate transformations in general relativity. Again the argument can be simplified if we make use of the rest frame, but in fact we will want to apply this argument to photons, which of course have no rest frame. We will therefore construct a derivation that makes no use of the rest frame. A general argument is given by Viatcheslav Mukhanov in **Physical Foundations of Cosmology** (Cambridge University Press, 2005), which I believe is generally a very good book. Nonetheless, his argument for the phase space invariance, on p. 358, appears to ignore all the complications that arise when the coordinate transformation changes the definition of time. If any of you find a way to justify his argument as written, please let me know.

We begin by examining the general problem of how to describe the density of things that move. At this point we can be very general, considering arbitrary "particles" that move in some arbitrary way through some arbitrary space described by some arbitrary coordinate system. We will, however, assume one important restriction: the velocity of the particles will be uniquely determined by their position and the time. Suppose the space is described by n coordinates, which we will call $\xi^1, \ldots, \xi^n$, and there is also a time coordinate t. For some purposes we will include t as one of the coordinates, in which case we will call them X^μ, with $X^0 = t$, $X^i = \xi^i$, for $i = 1$ to n. The particles will be labeled by an index α, and the α'th particle travels on a trajectory described with an arbitrary parameter λ:

$$\begin{aligned} \xi^i &= \xi^i_\alpha(\lambda) \\ t &= t_\alpha(\lambda)\ , \end{aligned} \tag{15}$$

or in a more compact notation,

$$X^\mu = X^\mu_\alpha(\lambda)\ . \tag{16}$$

We define a density $\rho(\xi^i, t)$ so that $\rho(\xi^i, t)\mathrm{d}^n\xi$ is the number of particles in a volume $\mathrm{d}^n\xi$ about ξ^i at time t.

(a) Show that ρ can be written as

$$\begin{aligned} \rho(\xi^i, t) &= \sum_\alpha \int \mathrm{d}\lambda\, \delta^n\big(\xi^i - \xi^i_\alpha(\lambda)\big)\, \delta\big(t - t_\alpha(\lambda)\big)\, \frac{\mathrm{d}t_\alpha}{\mathrm{d}\lambda} \\ &= \sum_\alpha \int \mathrm{d}\lambda\, \delta^{n+1}\big(X^\mu - X^\mu_\alpha(\lambda)\big)\, \frac{\mathrm{d}t_\alpha}{\mathrm{d}\lambda}\ . \end{aligned} \tag{17}$$

(b) Now consider an arbitrary change in coordinates, which might involve the time variable along with the other coordinates. Let the transformation be defined by the functions

$$X'^\mu = X'^\mu_c(X^\nu) \tag{18a}$$

and their inverse,

$$X^\mu = X^\mu_c(X'^\nu)\ , \tag{18b}$$

where I use the subscript c for the coordinate transformation. The trajectories in the new system are then given by

$$X'^\mu = X'^\mu_c(X^\nu_\alpha(\lambda))\ , \tag{19}$$

and the density function $\rho'(\xi'^i, t')$ is defined in terms of the primed coordinate trajectories by a formula analogous to Eq. (17). Show that the new density function $\rho'(\xi'^i, t')$ is related to $\rho(\xi^i, t)$ by

$$\rho'(\xi'^i, t') = \rho(\xi^i, t) \times \text{ Det} \left(\frac{\partial X^\mu_c}{\partial X'^\nu} \right) \left(\frac{\partial t'_c}{\partial t} + \frac{\partial t'_c}{\partial \xi^i} \frac{\mathrm{d}\xi^i}{\mathrm{d}t} \right) . \tag{20}$$

Here we used our assumption that the velocities depend only on ξ^i and t, so that the last factor, which depends on $\mathrm{d}\xi^i/\mathrm{d}t$, could be factored out of the sum and integral defining ρ.

Note that last factor in Eq. (20) differs from unity only when the transformation changes the equal-time hypersurfaces. In the method suggested here, this factor was found from the properties of delta functions. It is worth noting that in a more geometric approach, this factor would arise directly from the redefinition of the equal time hypersurfaces. For example, if two particles cross an equal-t hypersurface at the same value of the parameter λ, they will in general cross the equal-t' hypersurface at different values of λ, so their motion will cause their separation to be different on the two equal-time hypersurfaces. In the special relativity calculation of the previous problem, this space-dependent time offset was accounted for by the usual calculation of Lorentz contraction, which computes the distance between the two ends of the ruler measured at the same time in the new frame. In this problem, if one solves it by the direct computation of a Jacobian, the Jacobian must be computed for the transformation that maps the unprimed phase space coordinates on an equal-t hypersurface to the primed phase space coordinates that correspond to the same trajectories, but evaluated on an equal-t' hypersurface.

Some of you may wish to derive Eq. (20) by some other method, since there are many ways to derive it. If you wish to try a different method, it may help to be reminded of the following mathematical identities:

$$\text{Det} \left(\frac{\partial \xi^i_c}{\partial \xi'^j} \right) = \frac{\partial t'_c}{\partial t} \text{ Det} \left(\frac{\partial X^\mu_c}{\partial X'^\nu} \right) . \tag{21}$$

$$\text{Det} \left(\delta^i_j + u^i v_j \right) = 1 + u^i v_i . \tag{22}$$

$$\frac{\partial \xi'^i_c}{\partial \xi^j} \frac{\partial \xi^j_c}{\partial \xi'^k} = \delta^i{}_k - \frac{\partial \xi'^i_c}{\partial t} \frac{\partial t_c}{\partial \xi'^k} . \tag{23}$$

$$\frac{\partial t'_c}{\partial \xi^j} \frac{\partial \xi^j_c}{\partial \xi'^k} = - \frac{\partial t'_c}{\partial t} \frac{\partial t_c}{\partial \xi'^k} . \tag{24}$$

$$\frac{\partial t'_c}{\partial \xi^j} \frac{\partial \xi^j_c}{\partial t'} = 1 - \frac{\partial t'_c}{\partial t} \frac{\partial t_c}{\partial t'} . \tag{25}$$

Eq. (21) is Cramer's rule for the inverse of a matrix, making use of the fact that $\partial t'_c/\partial t$ is the 0–0 component of the inverse of the matrix $\partial X^\mu_c/\partial X'^\nu$. Eq. (22) is demonstrated by using $\mathrm{Det}\, M = \mathrm{Det}\,(O^{-1}MO)$, where we can choose O to be an orthogonal matrix that rotates v_i so that it points along one of the coordinate axes. The remaining three identities are special cases of the chain rule,

$$\frac{\partial X_c^\mu}{\partial X^\lambda}\frac{\partial X_c^\lambda}{\partial X^\nu} = \delta^\mu{}_\nu \ .$$

Now we are ready to apply Eq. (20) to the phase space problem. Here $n = 6$, with

$$\xi^1 = x^1 \ , \ \xi^2 = x^2 \ , \ \xi^3 = x^3 \ , \ \xi^4 = p_1 \ , \ \xi^5 = p_2 \ , \ \xi^6 = p_3 \ . \tag{26}$$

Consider a change of spacetime coordinates

$$x'^\mu = x_c'^\mu(x^\nu) \ , \tag{27}$$

so

$$\begin{aligned} t' &= x_c'^0(x^\nu) \equiv t_c'(x^\nu) \ , \\ x'^i &= x_c'^i(x^\nu) \ , \\ p_i' &= \frac{\partial x_c^\nu}{\partial x'^i} p_\nu \\ &= \frac{\partial x_c^j}{\partial x'^i} p_j + \frac{\partial t_c}{\partial x'^i} p_0(x^1, x^2, x^3, p_1, p_2, p_3, t) \ , \end{aligned} \tag{28}$$

where $p_0(x^1, x^2, x^3, p_1, p_2, p_3, t)$ is determined by Eq. (10) (from Problem 1).

(c) Now it's time for you to finish the calculation, showing that the phase space density $\mathcal{N}(x^i, p_i, t)$, as defined by Eq. (13) (in Problem 2), is invariant under general coordinate transformations. The recommended method is to use the general formula in Eq. (20), calculating the necessary derivatives from Eqs. (28), and making use of the Hamiltonian equations of motion (Eqs. (9) and (11)). You will probably find it helpful to be aware of the identities in Eqs. (21)–(25).

PROBLEM 4: SPECIFIC INTENSITY *(10 points)*

A quantity of interest to astronomers is the specific intensity I_ν, defined as the electromagnetic energy received by a detector per unit time, per unit detector area, per unit frequency interval, per unit solid angle. It is described, for example, in **Cosmological Physics**, by John A. Peacock (Cambridge University Press, 1999) at pp. 290 and 395. Show that, for each circular polarization of light, the specific intensity is related to the photon phase space density by

$$\mathcal{N}_\gamma = \frac{c^2}{(2\pi\hbar)^4}\frac{I_\nu}{\nu^3} \ . \tag{29}$$

(Suggestion: a key step is to express the momentum volume d^3p in terms of the solid angle $\mathrm{d}\Omega$, the frequency ν, and the frequency interval $\mathrm{d}\nu$.) Combined with the previous result, Eq. (29) guarantees that I_ν/ν^3 is Lorentz-invariant and fact invariant under general coordinate transformations.