MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.952: The Early Universe — May 8, 2009
Prof. Alan Guth

PROBLEM SET 5

DUE DATE: Thursday, May 14, 2009, at 5 pm.

The goal of these problems is to solidify our understanding of the techniques that Weinberg introduces in Chapter 5, including the derivation of the equations of motion, the relation between the Newtonian and synchronous gauges, and the adiabatic solutions to the perturbative equations.

PROBLEM 1: EINSTEIN EQUATIONS IN SYNCHRONOUS GAUGE *(15 points)*

To see what is involved in the equations of motion, derive the δR_{jk} Einstein equation in synchronous gauge. You can start with Weinberg's Eq. (5.1.13), using also Eqs. (5.1.16) and (5.1.21). Include the scalar functions A and B and also the tensor function D_{ij} in your expression for $h_{\mu\nu}$. To avoid getting excessively bogged down in algebra, we will leave out the vector modes. First find the tensor equation with indices j and k, and then extract the scalar equations for A and B.

One of the equations you will find should almost match Eq. (5.3.29), but you should actually find that

$$-16\pi G a^2 \partial_i \partial_j \pi^S = \partial_i \partial_j (A - a^2 \ddot{B} - 3a\dot{a}\dot{B}) \; ,$$

in analogy with the Newtonian gauge equation (5.3.20). Weinberg writes the formula without the derivatives. For Fourier modes with $\vec{q} \neq 0$, the equations are equivalent, but below we will be interested in zero wave number modes, so it will be worth knowing that Eq. (5.3.29) does not apply to them. It is actually obvious that Eq. (5.3.29) cannot apply to homogeneous solutions, since a homogeneous value for B makes no contribution to the metric. You will be able to extract a second scalar equation from the R_{jk} equation, which should match one of Weinberg's Eqs. (5.3.28)–(5.3.31).

PROBLEM 2: HOMOGENEOUS GAUGE TRANSFORMATIONS IN SYNCHRONOUS GAUGE *(10 points)*

In Section (V.4), as the first step in the construction of the adiabatic solutions, Weinberg constructs the most general gauge transformation in Newtonian gauge that preserve the conditions for Newtonian gauge and spatial homogeneity. You are asked to carry out the same exercise in synchronous gauge, finding the most general gauge transformation of the unperturbed solution that preserves spatial homogeneity and the synchronous gauge conditions. Show that the most general homogeneous solution that can be constructed in this way is described by two parameters, ϵ and ω_{kk}, with

$$A = 2H\epsilon - \frac{2}{3}\omega_{kk} \ ,$$

or something equivalent (depending on your choice of parameterization). B is irrelevant for homogeneous solutions, since it has no effect on the metric, but you should find the expressions for δp, $\delta\rho$, δu, and π^S, in analogy to Eqs. (5.4.8) and (5.4.9).

(Note that Weinberg constructs his argument based on gauge transformations described by the four-vector $\epsilon^\mu(x)$, as in Eqs. (5.3.5)–(5.3.7), without using the scalar/vector/tensor decomposition of Eq. (5.3.13). It is presumably possible to use the scalar/vector/tensor decomposition, but this decomposition ceases to be unique when the gauge function $\epsilon^\mu(x)$ does not approach zero at spatial infinity.)

PROBLEM 3: CONSTRUCTION OF ADIABATIC SOLUTIONS *(10 points)*

Continuing to follow the method Weinberg used for Newtonian gauge, try to construct the adiabatic solutions in synchronous gauge by allowing the gauge parameters of the homogeneous solution to vary sinusoidally with very long wavelengths. The goal is to construct a solution that obeys the gauge conditions and the constraint equations exactly, and obeys the dynamical equations in an approximation that becomes exact for asymptotically long wavelengths.

Recall that when Weinberg discussed the addition of spatial variations, he had to consider the constraint equations (5.3.20) and (5.3.21), which are satisfied trivially for homogeneous solutions. Here we have a similar situation with Eqs. (5.3.29) and (5.3.30), where (5.3.30) should really be written as $8\pi Ga(\bar\rho + \bar p)\partial_i\, \delta u = a\partial_i A$. Show that Eq. (5.3.30) is already satisfied for the homogeneous solution, and find an expression for $B(t)$ such that Eq. (5.3.29) is satisfied. There are are at least three ways that you can find $B(t)$. First, you can directly find a function $B(t)$ that satisfies Eq. (5.3.29). Second, you can use the gauge transformation equations (5.3.13) to find an expression for ϵ^S which assures that $\Delta F = 0$. Or third, you can come back to this after doing the next problem, in which you will learn to relate the synchronous gauge adiabatic solutions to the Newtonian gauge solutions.

PROBLEM 4: GAUGE EQUIVALENCE OF THE ADIABATIC SOLUTIONS IN SYNCHRONOUS AND NEWTONIAN GAUGES *(10 points)*

Starting with the adiabatic solutions in either of the two gauges (your choice), show how to carry out the procedure of Section V.3(C) to obtain the solutions in the other gauge.