MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.952: The Early Universe — February 11, 2009
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PROBLEM SET 1

DUE DATE: Wednesday, February 18, 2009

PROBLEM 1: A ZERO MASS DENSITY UNIVERSE— GENERAL RELATIVITY DESCRIPTION *(10 points)*

In this problem and the next we will explore the connections between special relativity and general relativity in the way that they describe an empty universe. In the limit of zero mass density the effects of gravity will vanish, so the general relativistic description of a Robertson-Walker universe must be equivalent to special relativity. The goal of these two problems is to see exactly how this happens. Historically, the empty universe model was proposed by the British astrophysicist Edward Arthur Milne as an alternative to general relativity, and is usually called the Milne universe.

These two problems will emphasize the notion that a coordinate system is nothing more than an arbitrary system of designating points in spacetime. A physical system might therefore look very different in two different coordinate systems, but the answer to any well-defined physical question must turn out the same regardless of which coordinate system is used in the calculation.

From the general relativity point of view, the model universe is described by the Robertson-Walker metric:

$$ds^2 = -c^2 dt^2 + a^2(t) \left\{ \frac{dr^2}{1 - Kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right\} .$$

The evolution of the model universe is governed by the usual Friedmann equation,

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3} G\rho - \frac{K}{a^2} ,$$

except that in this case the mass density term is to be set equal to zero.

(a) Since the mass density is zero, it is certainly less than the critical mass density, so the universe is open. We can then choose $K = -1$. Derive an explicit expression for the scale factor $a(t)$.

(b) Suppose that a light pulse is emitted by a comoving source at time t_e, and is received by a comoving observer at time t_o. Find the Doppler shift z.

(c) Consider a light pulse that leaves the origin at time t_e, traveling along a null trajectory, $ds^2 = 0$. Since the pulse is traveling in the radial direction (i.e., with $d\theta = d\phi = 0$), one has

$$cdt = a(t) \frac{dr}{\sqrt{1 - Kr^2}} \ .$$

Derive a formula for the trajectory $r(t)$ of the light pulse. You may find the following integral useful:

$$\int \frac{dr}{\sqrt{1 + r^2}} = \sinh^{-1} r \ .$$

(d) Use these results to express the redshift z in terms of the coordinate r of the observer. If you have done it right, your answer will be independent of t_e. (In the special relativity description that will follow, it will be obvious why the redshift must be independent of t_e.)

PROBLEM 2: A ZERO MASS DENSITY UNIVERSE— SPECIAL RELATIVITY DESCRIPTION *(10 points)*

In this problem we will describe the same model universe as in the previous problem, but we will use special relativity directly. We will therefore use an inertial coordinate system, rather than the comoving system of the previous problem. Please note, however, that in the usual case in which gravity is significant, there is no inertial coordinate system. Such a coordinate system exists only in the absence of gravity.

To distinguish the two systems, we will use primes to denote the inertial coordinates: (t', x', y', z'). Since the problem is spherically symmetric, we will also introduce polar inertial coordinates (r', θ', ϕ') which are related to the Cartesian inertial coordinates by the usual relations:

$$\begin{aligned} x' &= r' \sin\theta' \cos\phi' \\ y' &= r' \sin\theta' \sin\phi' \\ z' &= r' \cos\theta' \ . \end{aligned}$$

In terms of these polar inertial coordinates, the Minkowski metric of special relativity can be written as

$$ds^2 = -c^2 dt'^2 + dr'^2 + r'^2 \left(d\theta'^2 + \sin^2\theta' d\phi'^2 \right) \ .$$

For purposes of discussion we will introduce a set of comoving observers which travel along with the matter in the universe, following the Hubble expansion pattern.

(Although the matter has a negligible mass density, I will assume that enough of it exists to define a velocity at any point in space.) These trajectories must all meet at some spacetime point corresponding to the instant of the big bang, and we will take that spacetime point to be the origin of the coordinate system. Since there are no forces acting in this model universe, the comoving observers travel on lines of constant velocity (all emanating from the origin). The model universe is then confined to the future light-cone of the origin.

(a) The cosmic time variable t used in the previous problem can be defined as the time measured on the clocks of the comoving observers, starting at the instant of the big bang. Using this definition and your knowledge of special relativity, find the value of the cosmic time t for given values of the inertial coordinates— i.e., find $t(t', r')$. [Hint: first find the velocity of a comoving observer who starts at the origin and reaches the spacetime point (t', r', θ', ϕ').]

(b) Let us assume that angular coordinates have the same meaning in the two coordinate systems, so that $\theta = \theta'$ and $\phi = \phi'$. We will verify in part (d) below that this assumption is correct. Using this assumption, find the value of the comoving radial coordinate r in terms of the inertial coordinates— i.e., find $r(t', r')$. [Hint: consider an infinitesimal line segment which extends in the θ-direction, with constant values of t, r, and ϕ.] Draw a graph of the t'-r' plane, and sketch in lines of constant t and lines of constant r.

(c) Show that the radial coordinate r of the comoving system is related to the magnitude of the velocity in the inertial system by

$$r = \frac{v/c}{\sqrt{1 - v^2/c^2}} \ .$$

Suppose that a light pulse is emitted at the spatial origin ($r' = 0$, $t' =$ *anything*) and is received by another comoving observer who is traveling at speed v. With what redshift z is the pulse received? Express z as a function of r, and compare your answer to part (d) of the previous problem.

(d) Show that the metric of the Robertson-Walker comoving coordinate system can be derived from the Minkowski metric of special relativity, a fact which completely establishes the equivalence of the two descriptions. To begin, first write out the set of transformation equations, expressing t', r', θ', and ϕ' in terms of t, r, θ, and ϕ. (In case your GR is rusty, I offer the following hint: one way to continue is to consider an infinitesimal line segment described in the comoving system by its two endpoints: (t, r, θ, ϕ) and $(t + dt, r + dr, \theta + d\theta, \phi + d\phi)$. Then, to first order in the infinitesimal quantities, calculate the coordinate differences in the inertial coordinate system: dt', dr', $d\theta'$, and $d\phi'$. Use these quantities and the Minkowski metric to evaluate ds^2, and if you have made no mistakes you will recover the Robertson-Walker metric used in the previous problem.)

PROBLEM 3: LUMINOSITY DISTANCE VS. z *(10 points)*

On p. 42, Weinberg gives the general formula for luminosity distance d_L as a function of redshift z, for a Robertson-Walker universe whose mass density includes nonrelativistic matter (Ω_M), relativistic matter (Ω_R), and vacuum energy (Ω_Λ):

$$d_L(z) = \frac{1+z}{H_0 \Omega_K^{1/2}} \sinh \left[\Omega_K^{1/2} \int_{1/(1+z)}^{1} \frac{dx}{x^2 \sqrt{\Omega_\Lambda + \Omega_K x^{-2} + \Omega_M x^{-3} + \Omega_R x^{-4}}} \right] ,$$

where for each type of matter $\Omega_i \equiv \rho_i / \rho_{\rm cr}$, and $\Omega_K \equiv 1 - \Omega_\Lambda - \Omega_M - \Omega_R$. Derive this formula, filling in the steps that Weinberg left out. Start from the Friedmann equation,

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3} G\rho - \frac{K}{a^2} ,$$

and use the dependence on $a(t)$ appropriate to each type of matter ρ_i.

PROBLEM 4: VARIATION OF REDSHIFT WITH TIME *(10 points)*

On p. 13, Weinberg shows that the Hubble expansion rate at the time t_1, the time at which the light that we are now receiving with redshift z left its source, is given by

$$H(t_1) = (1+z) H_0 - \frac{dz}{dt_0} ,$$

where H_0 is the present Hubble expansion rate, and t_0 is the time of observation. The derivative dz/dt_0 refers to the rate of change of z for a given comoving object. To get some idea of the relevant numbers, suppose we model our universe as a matter-dominated flat universe, with $a(t) \propto t^{2/3}$. For that model, calculate dz/dt_0 as a function of z and H_0.

PROBLEM 5: TRANSLATION SYMMETRY IN ROBERTSON-WALKER UNIVERSES *(10 points)*

Consider a universe described by a Robertson-Walker metric with $K = +1$. Give a transformation of comoving space coordinates that leaves the metric unchanged, and that takes the point $\boldsymbol{x} = (0,0,r)$ into a point $\boldsymbol{x} = (0,0,r')$, with no change in the time. (Hint: Consider the three-dimensional space as a surface of a four-dimensional ball, construct this transformation as a rotation in four dimensions, and then express it in terms of the Robertson-Walker coordinates.) Also give the corresponding transformation for $K = -1$.