MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.952: The Early Universe — February 21, 2009
Prof. Alan Guth

PROBLEM SET 2

DUE DATE: Friday, February 27, 2009, at 5 pm.

PROBLEM 1: THE MANY COORDINATE SYSTEMS OF DE SITTER SPACE *(20 points)*

The Robertson-Walker universe resulting from pure vacuum energy is called de Sitter space, because it was first described in 1917 by the Dutch astronomer, Willem de Sitter. As discussed in Weinberg's book, on pp. 44–45, de Sitter described the space with a static metric, which fit well with the prevailing view of the time, which held that the universe was static. De Sitter space can also be described, however, as a flat ($K = 0$) Robertson–Walker metric with $a(t) \propto e^{Ht}$, where H is a constant related to the vacuum energy density. In fact, it can also be described by both open and closed Robsertson-Walker metrics. Because of the high degree of symmetry in de Sitter space, the equal-time surfaces can be chosen in different ways to allow the many different choices of coordinate systems.

Since inflationary cosmological models go through a phase which is very nearly de Sitter space, it is useful to pursue the geometry of de Sitter space more thoroughly. Probably the most illuminating description of de Sitter space was given by Erwin Schrödinger in his 1956 book, **Expanding Universes**. The description begins with the construction of a (4+1)-dimensional Minkowski space, for which we take the coordinates as (X, Y, Z, W, V), with metric

$$\mathrm{d}s^2 = \mathrm{d}X^2 + \mathrm{d}Y^2 + \mathrm{d}Z^2 + \mathrm{d}W^2 - \mathrm{d}V^2 \ . \tag{1}$$

(3+1)-dimensional de Sitter space can then be described as the hyperboloid defined by the constraint equation

$$X^2 + Y^2 + Z^2 + W^2 - V^2 = H^{-2} \ . \tag{2}$$

Formulated this way, one can see that de Sitter space has an O(4,1) symmetry group, with 10 parameters, the same size as the Poincaré group of symmetries of Minkowski space.

(a) To show that the space described by Eqs. (1) and (2) matches the Robertson–Walker flat description of de Sitter space, define the coordinates

$$\begin{aligned} t &= H^{-1} \ln\left[H(W+V)\right] \\ x &= e^{-Ht} X \\ y &= e^{-Ht} Y \\ z &= e^{-Ht} Z \ . \end{aligned} \tag{3}$$

Find the inverse of this transformation, expressing X, Y, Z, W, and V in terms of x, y, z, and t, using the constraint of Eq. (2). Find the metric in terms of the new coordinates, showing that it has the form of a Robertson–Walker flat universe. Note that the flat coordinate system actually covers only half of the full de Sitter space, as defined by Eqs. (1) and (2).

(b) To construct a Robertson–Walker closed coordinate system for de Sitter space, note that the intersection of the hyperboloid with the hypersurface $V = V_0 =$ constant is the surface of a sphere in 4 Euclidean dimensions,

$$X^2 + Y^2 + Z^2 + W^2 = V_0^2 + H^{-2} \ , \tag{4}$$

also known as S^4. This is exactly the form of the spatial slices of a Robertson–Walker closed universe. Use this fact to find the Robertson–Walker closed coordinates r, θ, ϕ, and t in terms of X, Y, Z, W, and V, and show that the metric has the claimed form. What part of the full space is covered by these coordinates?

(c) To construct the Robertson–Walker open coordinate system, note that the intersection of the hyperboloid with the hypersurface $W = W_0 =$ constant leads to the equation

$$X^2 + Y^2 + Z^2 - V^2 = H^{-2} - W_0^2 \ , \tag{5}$$

which for $W_0 > H^{-1}$ is in fact a description of a slice of an open Robertson-Walker coordinate system. Use this fact to find the Robertson–Walker open coordinates r, θ, ϕ, and t in terms of X, Y, Z, W, and V, and show that the metric has the claimed form. What part of the full space is covered by these coordinates?

(d) Finally, to construct the static coordinate system that de Sitter first used, let

$$\begin{aligned} V &= \sqrt{H^{-2} - r^2}\,\sinh Ht \\ W &= \sqrt{H^{-2} - r^2}\,\cosh Ht \ . \end{aligned} \tag{6}$$

Fill in the rest of the transformation, and show that the resulting metric has the form quoted by Weinberg. Does this coordinate system describe the entire manifold? If not, how would you describe the region that it does cover?

PROBLEM 2: THE TRANSITION FROM DECELERATION TO ACCELERATION (Weinberg, Assorted Problem #5, with addition) *(10 points)*

Suppose that $\Omega_M = 0.25$ and $\Omega_\Lambda = 0.75$, with Ω_R negligible. What is the redshift at which the expansion of the universe stopped accelerating and began to accelerate. *Additional question:* If the universe is 13.7 billion years old, how long ago did the acceleration begin?

PROBLEM 3: THE VIRIAL THEOREM WITH A HYPOTHETICAL FORCE LAW (Weinberg, Assorted Problem #5) *(10 points)*

Suppose that the gravitational potential energy of any pair of galaxies with separation r decreases as r^{-n} instead of r^{-1}. What combination of the mass of a virialized cluster of galaxies and the Hubble constant could be calculated from measurements of angular separations and velocity dispersions of its individual galaxies?

PROBLEM 4: TIME OF EMISSION OF LIGHT FROM A VERY DISTANT GALAXY *(10 points)*

At present, the highest spectroscopically measured redshift is that of a Lyman-α emitting galaxy discovered at the Subaru Telescope*, with a redshift of $z = 6.96$. In this problem you will calculate the age of the universe at the time the light that we are now receiving was emitted from this galaxy. Calculate first in a simple model of a flat universe dominated by nonrelativistic matter, with $a(t) \propto t^{2/3}$, taking the present age as 13.7 Gyr. Then calculate it for a realistic model of our universe, using the following parameters:†

Parameter	**WMAP 5-Year Recommended Fit**
$\boldsymbol{H_0}$	70.5 ± 1.3 km·s^{-1}·Mpc^{-1}
Baryonic matter $\boldsymbol{\Omega_b}$	0.0456 ± 0.0015
Dark matter $\boldsymbol{\Omega_{\rm dm}}$	0.228 ± 0.013
Vacuum energy $\boldsymbol{\Omega_\Lambda}$	0.726 ± 0.015
Relativistic matter $\boldsymbol{\Omega_R}$	8.4×10^{-5}

The quoted uncertainties are 1 σ. Note, however, that you are not asked to calculate the uncertainty of your answer. For the realistic model, you will presumably need to do a numerical integral.

---

* M. Iye et al., "A galaxy at a redshift $z = 6.96$," *Nature* **443**, 186 (2006) [astro-ph/0609393].

† G. Hinshaw et al., "Five-year Wilkinson Microwave Anisotropy Probe (WMAP) observations: Data processing, sky maps, and basic results," *Ap. J. Suppl.* **180**, 225-245 (2009) [arXiv:0803.0732v2 [astro-ph]], Table 7, WMAP+BAO+SN. The value for Ω_R is calculated using a cosmic microwave background temperature $T_\gamma = 2.725$ K, including photons and three species of effectively massless neutrinos with a temperature $T_\nu = (4/11)^{1/3} T_\gamma$. The value for T_γ comes from J.C. Mather et al., "Calibrator design for the COBE Far-Infrared Absolute Spectrophotometer (FIRAS)," *Ap. J.* **512,** 511 (1999) [astro-ph/9810373].