MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.952: Particle Physics of the Early Universe — February 27, 2009

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PROBLEM SET 1 SOLUTIONS

PROBLEM 1: A ZERO MASS DENSITY UNIVERSE— GENERAL RELATIVITY DESCRIPTION *(10 points)*

(a) To find the behavior of $a(t)$ with time in a zero mass density universe set $\rho = 0$ and $K = -1$ in the Friedmann equation. The equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{a^2} \quad \Longrightarrow \quad \dot{a}(t)^2 = 1 \; . \tag{1.1}$$

We choose the positive sign when we take the square root of the above equation, since we believe the universe is expanding and not contracting. Then

$$\mathrm{d}a = \mathrm{d}t,$$

so integration gives

$$\boxed{a(t) = t \; .} \tag{1.2}$$

The possible constant of integration in the above equation is fixed by the convention that the zero of time is chosen to be the instant when a vanishes.

(b) We know the expression for the cosmological redshift is just

$$1 + z = \frac{a(t_o)}{a(t_e)} \; . \tag{1.3}$$

Using $a(t) = t$, this gives

$$\boxed{z = \frac{t_o}{t_e} - 1 \; .} \tag{1.4}$$

(c) We find the trajectory of the light pulse by solving

$$\mathrm{d}t = a(t)\frac{\mathrm{d}r}{\sqrt{1+r^2}} \tag{1.5}$$

for r as a fuction of t. Using $a(t) = t$, we write the above expression as

$$\frac{\mathrm{d}t}{t} = \frac{\mathrm{d}r}{\sqrt{1+r^2}} \tag{1.6}$$

and then integrate from the time of emission t_e to the time of observation t_o:

$$\int_{t_e}^{t_o} \frac{\mathrm{d}t'}{t'} = \int_0^r \frac{\mathrm{d}r'}{\sqrt{1+r'^2}} \quad \Longrightarrow \quad \ln(t_o/t_e) = \sinh^{-1} r \; . \tag{1.7}$$

Solving this for r gives

$$r = \sinh\left(\ln(t_o/t_e)\right) \; . \tag{1.8}$$

Remembering that

$$\sinh\theta = \frac{e^\theta - e^{-\theta}}{2} \; , \tag{1.9}$$

the expression can be rewritten as

$$\boxed{r = \frac{t_o/t_e - t_e/t_o}{2} = \frac{(t_o/t_e)^2 - 1}{2(t_o/t_e)} \; .} \tag{1.10}$$

(d) Defining

$$y \equiv 1 + z = \frac{a(t_o)}{a(t_e)} = \frac{t_o}{t_e} \; , \tag{1.11}$$

the result from part (c) becomes

$$r = \frac{y^2 - 1}{2y} \quad \Longrightarrow \quad y^2 - 2yr - 1 = 0 \; , \tag{1.12}$$

which implies that

$$y = r \pm \sqrt{r^2 + 1} \; . \tag{1.13}$$

Only the positive root is valid, since the negative root would give a physically meaningless negative value for t_o/t_e. (*Side comment:* Spurious solutions to quadratic equations often have a physical interpretation as the solution to a closely related physical problem, but here that does not seem to be the case. The spurious solution corresponds to a mathematical solution to Eq. (1.7) in which $\int \mathrm{d}t'/t'$ is integrated around the singularity at $t' = 0$ in the complex t' plane, so that the integral acquires an imaginary part $\pm i\pi$. The integral over r' on the right-hand side can acquire a matching imaginary part by distorting the contour of integration to encircle the branch point of the integrand at $r' = i$.) Thus,

$$\boxed{1 + z = r + \sqrt{r^2 + 1} \; .} \tag{1.14}$$

The fact that z depends only on r, and not t_e, is a consequence of the fact that there is no gravity in this problem. There is no force acting on the comoving observers, so they each move at a constant velocity as seen from the inertial Minkowski coordinate system. Thus, the redshift between any two observers cannot change with time.

PROBLEM 2: A ZERO MASS DENSITY UNIVERSE— SPECIAL RELATIVITY DESCRIPTION *(10 points)*

(a) Since there is no gravitational field, the comoving observers move at a constant velocity in the inertial frame of reference (described by coordinates t', r', θ', and ϕ'). Since the comoving observers all start at the origin of the coordinate system, each comoving observer travels on a trajectory $r' = vt'$, where

$$v = r'/t' \tag{2.1}$$

will have a different value for different comoving observers. The cosmic time t is defined to be the proper time as measured by comoving observers, so from the point of view of the inertial frame t is measured on clocks that are running slowly by a factor of $\gamma(v)$:

$$t = t'/\gamma(v) = t'\sqrt{1-v^2} = t'\sqrt{1-\frac{r'^2}{t'^2}}\ , \tag{2.2}$$

or

$$\boxed{t = \sqrt{t'^2 - r'^2}\ .} \tag{2.3}$$

Thus, t is just the Lorentz-invariant separation of (t', r') from the origin. Notice that since v is constant the comoving observers are also inertial observers in the special relativistic sense.

(b) The Robertson–Walker metric for this case is given by

$$ds^2 = -dt^2 + t^2\left\{\frac{dr^2}{1+r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right\}\ , \tag{2.4}$$

and the Minkowski metric has the form

$$ds^2 = -dt'^2 + dr'^2 + r'^2\left(d\theta'^2 + \sin^2\theta' d\phi'^2\right)\ . \tag{2.5}$$

Since we have assumed that $\theta' = \theta$ and $\phi' = \phi$, the angular pieces of the metrics match only if $r'^2 = r^2t^2$, so

$$\boxed{r = \frac{r'}{t} = \frac{r'}{\sqrt{t'^2 - r'^2}}\ .} \tag{2.6}$$

To sketch lines of constant t in the r'-t' plane, note that Eq. (2.3) can be rewritten as

$$t' = \sqrt{t^2 + r'^2}\ , \tag{2.7}$$

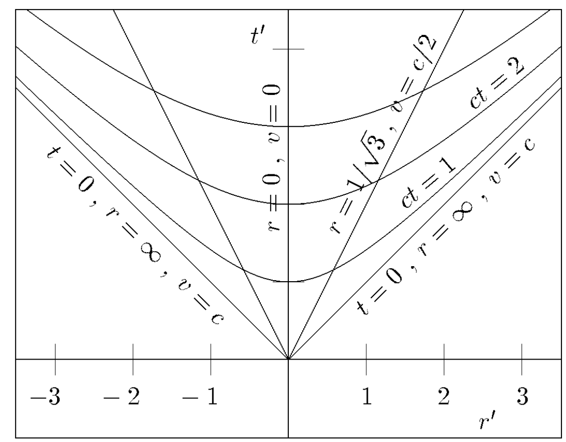
which for a fixed value of t describes a hyperbola. Each value of t gives a different hyperbola, and $t = 0$ gives the degenerate hyperbola $t' = |r'|$. To sketch lines of constant r, we can solve Eq. (2.6) for r'/t', finding

$$v = \frac{r'}{t'} = \frac{r}{\sqrt{1+r^2}}\ , \tag{2.8}$$

or

$$t' = \frac{\sqrt{1+r^2}}{r}\, r'\ . \tag{2.9}$$

Thus the lines of constant r are straight lines in the r'-t' plane. Note that as $r \to \pm\infty$, the slope approaches ± 1:



(c) We have shown in Eq. (2.8) that

$$v = \frac{r}{\sqrt{1+r^2}}\ ,$$

so all that remains is to calculate the redshift. The redshift in special relativity is given by

$$1 + z = \sqrt{\frac{1+v}{1-v}}\ . \tag{2.10}$$

Substituting the previous expression for v, one finds

$$1 + z = \sqrt{\frac{1 + \frac{r}{\sqrt{1+r^2}}}{1 - \frac{r}{\sqrt{1+r^2}}}} = \sqrt{\frac{\sqrt{1+r^2}+r}{\sqrt{1+r^2}-r}}\ . \tag{2.11}$$

The expression simplifies dramatically if one multiplies the numerator and denominator by $\sqrt{\sqrt{1+r^2}+r}$, yielding

$$1+z = \sqrt{\frac{(\sqrt{r^2+1}+r)(\sqrt{r^2+1}+r)}{(\sqrt{1+r^2}-r)(\sqrt{1+r^2}+r)}} = \boxed{r+\sqrt{1+r^2}\ .} \tag{2.12}$$

As expected, this agrees with the redshift found in part (d) of the previous problem.

(d) We have the following transformation equations:

$$\boxed{\begin{aligned} t &= \sqrt{t'^2 - r'^2} \\ r &= \frac{r'}{\sqrt{t'^2 - r'^2}} \\ \theta &= \theta' \\ \phi &= \phi'\ , \end{aligned}} \tag{2.13}$$

which can be inverted to give

$$\boxed{\begin{aligned} t' &= t\sqrt{1+r^2} \\ r' &= t\, r \\ \theta' &= \theta \\ \phi' &= \phi\ . \end{aligned}} \tag{2.14}$$

We thus find that, for an infinitesimal change in the coordinates,

$$\begin{aligned} \mathrm{d}t' &= \sqrt{1+r^2}\mathrm{d}t + \frac{rt}{\sqrt{1+r^2}}\mathrm{d}r \\ \mathrm{d}r' &= t\,\mathrm{d}r + r\,\mathrm{d}t \\ \mathrm{d}\theta' &= \mathrm{d}\theta \\ \mathrm{d}\phi' &= \mathrm{d}\phi\ . \end{aligned} \tag{2.15}$$

Finally, we substitute these expressions into the Minkowski metric of Eq. (2.5):

$$\begin{aligned} \mathrm{d}s^2 &= -\mathrm{d}t'^2 + \mathrm{d}r'^2 + r'^2(\mathrm{d}\theta'^2 + \sin^2\theta'\,\mathrm{d}\phi'^2) \\ &= -\left[\mathrm{d}t^2(1+r^2) + \frac{r^2\,t^2}{1+r^2}\mathrm{d}r^2 + 2rt\,\mathrm{d}r\,\mathrm{d}t\right] \\ &\quad + \left[t^2\mathrm{d}r^2 + r^2\mathrm{d}t^2 + 2rt\,\mathrm{d}r\,\mathrm{d}t\right] + t^2\,r^2[\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2] \\ &= -\mathrm{d}t^2 + t^2\left\{\frac{\mathrm{d}r^2}{1+r^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right\}\ , \end{aligned} \tag{2.16}$$

which agrees with the Robertson–Walker metric as shown in Eq. (2.4).

DISCUSSION OF THE ZERO MASS DENSITY UNIVERSE:

The two problems above demonstrate how the general relativistic description of cosmology can reduce to special relativity when gravity is unimportant, but it gives a misleading picture of the big-bang singularity which is worth discussing.

First, we should keep in mind that the mass density of the universe increases as we look backward in time. So, if we lived in a universe with a negligible value of Ω at the present time, then such a universe could be well-described at present by the empty Milne universe. Nonetheless, the universe would not be described by the Milne universe back to the singularity, as at early times the mass density would not be negligible. Thus, no matter how small the value of Ω today, as long as it is nonzero, the Milne universe description of $t \approx 0$ can be qualitatively different from a more realistic model.

In particular, the behavior $a(t) = t$ (for $K = -1$) differs from a realistic model in two important ways. First, for this special case the Riemann curvature tensor vanishes, as must be the case if the spacetime is equivalent to Minkowski space. One can check the usual Robertson–Walker equations to make sure that this is the case. Second, because the integral

$$\int_0^t \frac{\mathrm{d}t'}{a(t')}$$

diverges at the lower limit, the particle horizon in the Milne model is infinite. All particles are visible from the earliest times, a fact which is obvious in the Minkowski space description, where all comoving worldlines originate at the origin of the coordinate system.

PROBLEM 3: LUMINOSITY DISTANCE VS. z *(10 points)*

The Robertson–Walker metric,

$$ds^2 = -dt^2 + a^2(t)\left\{\frac{dr^2}{1-Kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\} , \tag{3.1}$$

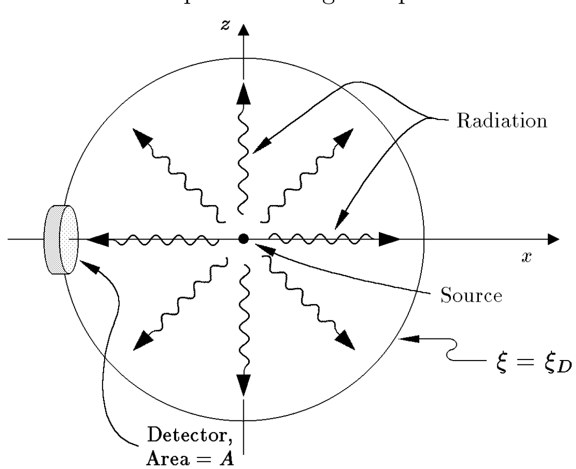
can be rewritten by defining

$$r = S_K(\xi) \equiv \begin{cases} \sin\xi & \text{if } K = 1 \\ \xi & \text{if } K = 0 \\ \sinh\xi & \text{if } K = -1 \end{cases} , \tag{3.2}$$

which gives

$$ds^2 = -dt^2 + a^2(t)\left\{d\xi^2 + S_K^2(\xi)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right\} . \tag{3.3}$$

To find the energy flux hitting a detector at radial coordinate ξ_D relative to a source, we need to consider the total power hitting the sphere at coordinate radius ξ_D:



The power hitting the sphere is given by

$$P = \frac{L}{(1+z)^2} , \tag{3.4}$$

where L is the absolute luminosity. One power of $1+z$ is due to the redshifting of the photons, and one power is due to the decrease in their rate of arrival. The energy flux is then

$$\ell = \frac{P}{\text{Area}} = \frac{L}{4\pi a^2(t_o) S_K^2(\xi_D)(1+z)^2} , \tag{3.5}$$

where t_o is the time of observation. The luminosity distance is defined by

$$d_L(z) = \sqrt{\frac{L}{4\pi\ell}} , \tag{3.6}$$

so the only remaining task is to find ξ_D in terms of z and other parameters. By setting $ds^2 = 0$ to follow the radial light pulses, we see that

$$\xi_D = \int_{t_e}^{t_o} \frac{dt'}{a(t')} , \tag{3.7}$$

where t_e is the time at which the light was emitted. Changing the variable of integration to

$$x \equiv \frac{a(t)}{a(t_o)} = \frac{1}{1+z(t)} , \tag{3.8}$$

where $z(t)$ is the redshift of light emitted at time t, the integral can be rewritten as

$$\xi_D = \frac{1}{a(t_o)} \int_{1/(1+z)}^{1} \frac{dx}{x\,\dot{x}} . \tag{3.9}$$

$\dot{x}$ can be evaluated using the Friedmann equation, supplemented by the conditions that $\rho_\Lambda = \text{const}$, $\rho_M \propto a^{-3}(t)$, and $\rho_R \propto a^{-4}(t)$. So

$$\begin{aligned} H^2 = \left(\frac{\dot{x}}{x}\right)^2 &= \frac{8\pi}{3}G\rho - \frac{K}{a^2} \\ &= H_o^2\left\{\frac{\rho}{\rho_c} + \frac{\Omega_K}{x^2}\right\} \\ &= H_o^2\left\{\Omega_\Lambda + \frac{\Omega_M}{x^3} + \frac{\Omega_R}{x^4} + \frac{\Omega_K}{x^2}\right\} , \end{aligned} \tag{3.10}$$

where Ω_Λ, Ω_M, and Ω_R are the contributions to Ω from vacuum energy, nonrelativistic matter, and relativistic matter, respectively, at the time of observation, and

$$\Omega_K = -\frac{K}{a^2(t_o)\, H_o^2} . \tag{3.11}$$

Applying Eq. (3.10) at $t = t_o$, one sees that

$$\Omega_\Lambda + \Omega_M + \Omega_R + \Omega_K = 1 \; . \tag{3.12}$$

Combining Eqs. (3.9) and (3.10), and then (3.11),

$$\begin{aligned} \xi_D &= \frac{1}{a(t_o)H_o} \int_{1/(1+z)}^{1} \frac{\mathrm{d}x}{x^2\sqrt{\Omega_\Lambda + \Omega_M x^{-3} + \Omega_R x^{-4} + \Omega_K x^{-2}}} \\ &= \sqrt{\frac{\Omega_K}{-K}} \int_{1/(1+z)}^{1} \frac{\mathrm{d}x}{x^2\sqrt{\Omega_\Lambda + \Omega_M x^{-3} + \Omega_R x^{-4} + \Omega_K x^{-2}}} \; . \end{aligned} \tag{3.13}$$

From Eqs. (3.5) and (3.6) one has

$$\boxed{d_L(z) = (1+z)\, a(t_o)\, S_K(\xi_D) \; ,} \tag{3.14}$$

so putting it all together we have

$$\boxed{\begin{aligned} d_L(z) = &\frac{1+z}{H_o}\sqrt{\frac{-K}{\Omega_K}} \\ &\times S_K\left\{\sqrt{\frac{\Omega_K}{-K}} \int_{1/(1+z)}^{1} \frac{\mathrm{d}x}{x^2\sqrt{\Omega_\Lambda + \Omega_M x^{-3} + \Omega_R x^{-4} + \Omega_K x^{-2}}}\right\} \; . \end{aligned}} \tag{3.15}$$

For $K = -1$ this agrees exactly with Weinberg's Eq. (1.5.45). The form shown here expresses the answer in terms of explicitly real functions for $K = 0$ and $K = 1$ as well, while Weinberg left these cases to be found by analytic continuation in Ω_K.

PROBLEM 4: VARIATION OF REDSHIFT WITH TIME *(10 points)*

There are at least two approaches to this problem, one of which is to accept the formula stated in the problem, so

$$\frac{dz}{dt_0} = (1+z)H_0 - H(t_1) \; . \tag{4.1}$$

Then the only task is to express $H(t_1)$ in terms of z and H_0. t_1 can be related to the redshift by

$$1 + z = \frac{a(t_0)}{a(t_1)} = \left(\frac{t_0}{t_1}\right)^{2/3} , \tag{4.2}$$

so

$$t_1 = \frac{t_0}{(1+z)^{3/2}} \; . \tag{4.3}$$

Then

$$\begin{aligned} H(t_1) &= \frac{\dot a(t_1)}{a(t_1)} = \frac{2}{3t_1} \\ &= (1+z)^{3/2}\frac{2}{3t_0} = (1+z)^{3/2} H_0 \; . \end{aligned} \tag{4.4}$$

Finally, using Eqs. (4.1) and (4.4),

$$\boxed{\frac{dz}{dt_0} = (1+z)\left[1 - \sqrt{1+z}\,\right] H_0 \; .} \tag{4.5}$$

A second approach would be to calculate $z(r, t_0)$ directly, and then differentiate it. For definiteness, let

$$a(t) = Bt^{2/3} \; , \tag{4.6}$$

where B is a constant. The time of emission t_1 will be related to r by

$$r = \int_{t_1}^{t_0} \frac{\mathrm{d}t'}{a(t')} = \int_{t_1}^{t_0} \frac{\mathrm{d}t'}{Bt'^{2/3}} = \frac{3}{B}\left(t_0^{1/3} - t_1^{1/3}\right) \; , \tag{4.7}$$

so

$$t_1 = \left(t_0^{1/3} - \frac{Br}{3}\right)^3 \; , \tag{4.8}$$

and then

$$1 + z = \frac{t_0^{2/3}}{t_1^{2/3}} = \frac{t_0^{2/3}}{\left(t_0^{1/3} - \frac{Br}{3}\right)^2} \; . \tag{4.9}$$

Differentiating the expression above,

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}t_0} &= \frac{2}{3}\frac{t_0^{-1/3}}{\left(t_0^{1/3} - \frac{Br}{3}\right)^2} - \frac{2}{3}\frac{1}{\left(t_0^{1/3} - \frac{Br}{3}\right)^3} \\ &= \frac{2}{3t_0}\left[1 + z - (1+z)^{3/2}\right] \\ &= \boxed{H_0(1+z)\left[1 - \sqrt{1+z}\,\right] \; .} \end{aligned} \tag{4.10}$$

PROBLEM 5: TRANSLATION SYMMETRY IN ROBERTSON-WALKER UNIVERSES *(10 points)*

I (AHG) found the wording of this problem ambiguous, because it was not clear whether it referred to the usual polar coordinate form of the Robertson–Walker metric, Eq. (1.1.11), or the quasi-Cartesian form of Eq. (1.1.9). For purposes of the problem set, both interpretations will be accepted. I initially assumed that Weinberg was referring to the polar form, since that is the traditional form of the Robertson–Walker metric, and because it was suggested by the use of the coordinate values r and r'. In hindsight, however, I am sure that Weinberg intended the problem to be worked in the quasi-Cartesian coordinates, because it is much simpler in that form, and the use of the boldface vector $\boldsymbol{x}$ suggests this form. Here I will show both solutions, starting with the polar coordinate formulation.

The Robertson–Walker closed universe can be described simply by embedding it in one extra space dimension, so that it becomes the three-dimensional surface of a sphere in four Euclidean dimensions. Without loss of generality the sphere can be taken as a unit sphere, with actual size described by the scale factor, which multiplies the coordinate dimensions. If we use coordinates (w, x, y, z) for the 4D embedding space, with the physical subspace described by

$$w^2 + x^2 + y^2 + z^2 = 1 \,, \tag{5.1}$$

then the Robertson–Walker polar coordinates can be described by

$$\begin{aligned} w &= \sqrt{1-r^2} \\ x &= r\,\sin\theta\,\cos\phi \\ y &= r\,\sin\theta\,\sin\phi \\ z &= r\,\cos\theta \,. \end{aligned} \tag{5.2}$$

It will also be useful to define

$$r = \sin\psi \,, \tag{5.3}$$

where ψ is the angle of the point (w, x, y, z) from the w-axis.

The problem asks us to find a coordinate transformation that takes the point $(0, 0, r)$ into the point $(0, 0, r')$. To simplify the notation, I will reserve the use of primes to indicate the coordinate transformation — it will be described by defining a primed coordinate system in terms of an unprimed one. I will therefore reword the original question, seeking a coordinate transformation that takes the point $(0, 0, r_1)$ into the point $(0, 0, r_2)$.

The transformation is simple in terms of the 4D coordinates, where it is just a rotation. Defining

$$r_1 = \sin\psi_1 \,, \quad r_2 = \sin\psi_2 \,, \tag{5.4}$$

the desired coordinate transformation should rotate in the w-z plane by an angle

$$\alpha = \psi_2 - \psi_1 \,. \tag{5.5}$$

Thus,

$$\begin{aligned} w' &= w\cos\alpha - z\sin\alpha \\ z' &= z\cos\alpha + w\sin\alpha \\ x' &= x \\ y' &= y \,, \end{aligned} \tag{5.6}$$

where

$$\begin{aligned} \sin\alpha &= \sin\psi_2\,\cos\psi_1 - \sin\psi_1\,\cos\psi_2 = r_2\sqrt{1-r_1^2} - r_1\sqrt{1-r_2^2} \\ \cos\alpha &= \cos\psi_2\,\cos\psi_1 + \sin\psi_2\,\sin\psi_1 = \sqrt{1-r_2^2}\,\sqrt{1-r_1^2} + r_2\,r_1 \,. \end{aligned} \tag{5.7}$$

The point $(0, 0, r_1)$ corresponds to $(w, x, y, z) = (\sqrt{1-r_1^2}, 0, 0, r_1)$, and from Eqs. (5.6) and (5.7), one can verify that this point is mapped to $(w', z', x', y') = (\sqrt{1-r_2^2}, 0, 0, r_2)$, as intended.

The primed 4D coordinates are related to (r', θ', ϕ') as in Eq. (5.2), so

$$\begin{aligned} w' &= \sqrt{1-r'^2} \\ x' &= r'\,\sin\theta'\,\cos\phi' \\ y' &= r'\,\sin\theta'\,\sin\phi' \\ z' &= r'\,\cos\theta' \,. \end{aligned} \tag{5.8}$$

Therefore, using the first of Eqs. (5.6), one finds that

$$\sqrt{1-r'^2} = \sqrt{1-r^2}\cos\alpha - r\cos\theta\sin\alpha \,, \tag{5.9}$$

from which one finds

$$\boxed{r' = \sqrt{1 - \left[\sqrt{1-r^2}\cos\alpha - r\cos\theta\sin\alpha\right]^2} \,.} \tag{5.10}$$

Since x and y are preserved by the transformation, the angle in the x-y plane is preserved, so

$$\boxed{\phi' = \phi \,.} \tag{5.11}$$

The invariance of x and y also implies that $r' \sin\theta' = r\sin\theta$, so

$$\boxed{\sin\theta' = \frac{r\sin\theta}{\sqrt{1-\left[\sqrt{1-r^2}\cos\alpha - r\cos\theta\sin\alpha\right]^2}}\ .} \tag{5.12}$$

Alternatively, one can find an equation for $\cos\theta'$ by using the z' equation:

$$\boxed{\cos\theta' = \frac{r\cos\theta\cos\alpha + \sqrt{1-r^2}\sin\alpha}{\sqrt{1-\left[\sqrt{1-r^2}\cos\alpha - r\cos\theta\sin\alpha\right]^2}}\ .} \tag{5.13}$$

One can verify that the two expressions above are consistent with $\sin^2\theta' + \cos^2\theta' = 1$, so Eq. (5.12) could have been derived from Eq. (5.13). Thus, the boxed equations above define the transformation, but only one of Eqs. (5.12) and (5.13) is needed.

You were not asked to do so, but since the transformation of Eqs. (5.10)-(5.13) is supposed to leave the metric invariant, it seems appropriate to check explicitly that this is true. The calculation is very complicated, however, so one would not want to approach it without the help of a computer algebra program. Using such help, I found the following partial derivatives:

$$\begin{aligned}
\frac{\partial r'}{\partial r} &= \frac{r\left[\cos^2\alpha + \left(\frac{\sqrt{1-r^2}}{r} - \frac{r}{\sqrt{1-r^2}}\right)\sin\alpha\cos\alpha\cos\theta - \sin^2\alpha\cos^2\theta\right]}{\sqrt{\left[r\cos\alpha + \sqrt{1-r^2}\sin\alpha\cos\theta\right]^2 + \sin^2\alpha\,\sin^2\theta}} \\
\frac{\partial r'}{\partial\theta} &= \frac{r\sin\alpha\sin\theta\,\left(r\sin\alpha\cos\theta - \sqrt{1-r^2}\cos\alpha\right)}{\sqrt{\left[r\cos\alpha + \sqrt{1-r^2}\sin\alpha\cos\theta\right]^2 + \sin^2\alpha\,\sin^2\theta}} \\
\frac{\partial\theta'}{\partial r} &= \frac{\sin\alpha\,\sin\theta}{\sqrt{1-r^2}\left\{\left[r\cos\alpha + \sqrt{1-r^2}\sin\alpha\cos\theta\right]^2 + \sin^2\alpha\,\sin^2\theta\right\}} \\
\frac{\partial\theta'}{\partial\theta} &= \frac{r\left[2r^2\cos^2\alpha\cos\theta + r\sqrt{1-r^2}\sin\alpha\cos\alpha(\cos^2\theta+1) + \cos\theta(\sin^2\alpha - r^2)\right]}{\left(r\cos\alpha\cos\theta + \sqrt{1-r^2}\sin\alpha\right)\left\{\left[r\cos\alpha + \sqrt{1-r^2}\sin\alpha\cos\theta\right]^2 + \sin^2\alpha\,\sin^2\theta\right\}}
\end{aligned} \tag{5.14}$$

Next we express $\mathrm{d}r'$ and $\mathrm{d}\theta'$ in terms of the unprimed quantities:

$$\begin{aligned}
\mathrm{d}r' &= \frac{\partial r'}{\partial r}\mathrm{d}r + \frac{\partial r'}{\partial\theta}\mathrm{d}\theta \\
\mathrm{d}\theta' &= \frac{\partial\theta'}{\partial r}\mathrm{d}r + \frac{\partial\theta'}{\partial\theta}\mathrm{d}\theta
\end{aligned} \tag{5.15}$$

Recalling that $\phi' = \phi$ and that $r'\sin\theta' = r\sin\theta$, and again making heavy use of the computer algebra program, one can use Eqs. (5.14) and (5.15) to show that the spatial metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}r'^2}{1-r'^2} + r'^2\left(\mathrm{d}\theta'^2 + \sin^2\theta'\,\mathrm{d}\phi'^2\right) \tag{5.16}$$

can be rewritten as

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1-r^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\ , \tag{5.17}$$

which verifies that the metric is indeed invariant under the transformation described by Eqs. (5.10)–(5.13).

For the open universe case, one starts by introducing a 4D embedding space (w, x, y, z) with a pseudo-Euclidean metric

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - \mathrm{d}w^2\ . \tag{5.18}$$

The metric is of course equivalent to the Minkowski metric, but we should remember that w has no physical connection to time. The Robertson–Walker spatial slice is described by the subspace satisfying

$$x^2 + y^2 + z^2 - w^2 = -1\ , \tag{5.19}$$

and the Robertson–Walker polar coordinates are defined by

$$\begin{aligned}
w &= \sqrt{1+r^2} \\
x &= r\,\sin\theta\,\cos\phi \\
y &= r\,\sin\theta\,\sin\phi \\
z &= r\,\cos\theta\ ,
\end{aligned} \tag{5.20}$$

where this time we define

$$r = \sinh\psi\ . \tag{5.21}$$

This time the transformation will be a pseudo-rotation in the w-z plane, which in the context of the Lorentz group would be called a boost. Thus,

$$\begin{aligned}
w' &= w\cosh\alpha + z\sinh\alpha \\
z' &= z\cosh\alpha + w\sinh\alpha \\
x' &= x \\
y' &= y\ ,
\end{aligned} \tag{5.22}$$

where α can be expressed in terms of the ψ's by Eq. (5.5), so

$$\begin{aligned} \sinh\alpha &= \sinh\psi_2\,\cosh\psi_1 - \sinh\psi_1\,\cosh\psi_2 = r_2\sqrt{1+r_1^2} - r_1\sqrt{1+r_2^2} \\ \cosh\alpha &= \cosh\psi_2\,\cosh\psi_1 - \sinh\psi_2\,\sinh\psi_1 = \sqrt{1+r_2^2}\,\sqrt{1+r_1^2} - r_2\,r_1\ . \end{aligned} \tag{5.23}$$

This time the point $(0,0,r_1)$ corresponds to $(w,x,y,z) = (\sqrt{1+r_1^2},0,0,r_1)$, and from Eqs. (5.22) and (5.23), one can verify that this point is mapped to $(w',z',x',y') = (\sqrt{1+r_2^2},0,0,r_2)$, as intended.

The primed 4D coordinates are related to (r',θ',ϕ') as in Eq. (5.20), so

$$\begin{aligned} w' &= \sqrt{1+r'^2} \\ x' &= r'\,\sin\theta'\,\cos\phi' \\ y' &= r'\,\sin\theta'\,\sin\phi' \\ z' &= r'\,\cos\theta'\ . \end{aligned} \tag{5.24}$$

Therefore

$$\sqrt{1+r'^2} = \sqrt{1+r^2}\cosh\alpha + r\cos\theta\sinh\alpha\ , \tag{5.25}$$

from which one finds

$$\boxed{r' = \sqrt{\left[\sqrt{1+r^2}\cosh\alpha + r\cos\theta\sinh\alpha\right]^2 - 1}\ .} \tag{5.26}$$

Again the fact that x and y are unchanged implies that

$$\boxed{\phi' = \phi} \tag{5.27}$$

and that $r'\sin\theta' = r\sin\theta$, so

$$\boxed{\sin\theta' = \frac{r\sin\theta}{\sqrt{\left[\sqrt{1+r^2}\cosh\alpha + r\cos\theta\sinh\alpha\right]^2 - 1}}\ .} \tag{5.28}$$

As in the previous case, one can use the z' equation to obtain a relation for $\cos\theta'$:

$$\boxed{\cos\theta' = \frac{r\cos\theta\cosh\alpha + \sqrt{1+r^2}\sinh\alpha}{\sqrt{\left[\sqrt{1+r^2}\cosh\alpha + r\cos\theta\sinh\alpha\right]^2 - 1}}\ .} \tag{5.29}$$

Alternatively, the question may have been intended to refer to the Robertson–Walker coordinate system with quasi-Cartesian coordinates, with spatial metric

$$ds^2 = a^2\left[d\vec{x}^2 + K\frac{(\vec{x}\cdot d\vec{x})^2}{1-K\vec{x}^2}\right]\ . \tag{5.30}$$

These coordinates can be embedded in a 4D Euclidean or pseudo-Euclidean space (w,x,y,z) by adding the redundant coordinate w, given by

$$w = \sqrt{1-K(x^2+y^2+z^2)}\ . \tag{5.31}$$

The metric in the 4D space is

$$ds^2 = dx^2 + dy^2 + dz^2 + K^{-1}dw^2\ , \tag{5.32}$$

and the constraint is

$$x^2+y^2+z^2+K^{-1}w^2 = K^{-1}\ , \tag{5.33}$$

where $K=1$ for a closed universe and $K=-1$ for an open universe.

Considering first the closed universe case $K=1$, the point $(x,y,z) = (0,0,r_1)$ corresponds to $(w,x,y,z) = (\sqrt{1-r_1^2},0,0,r_1)$, and $(0,0,r_2)$ corresponds to $(\sqrt{1-r_2^2},0,0,r_2)$. Thus, the angle ψ from the w-axis is given by $r=\sin\psi$, so the first point is carried into the second by a rotation in the w-z plane by an angle $\alpha = \psi_2 - \psi_1$, as in Eq. (5.5). The rotation is given by Eq. (5.6), so we can see immediately that

$$\boxed{\begin{aligned} x' &= x \\ y' &= y\ , \end{aligned}} \tag{5.34}$$

and then

$$\begin{aligned} z' &= z\cos\alpha + w\sin\alpha \\ &= \boxed{z\cos\alpha + \sqrt{1-(x^2+y^2+z^2)}\sin\alpha\ .} \end{aligned} \tag{5.35}$$

On can use Eqs. (5.34) and (5.35) to calculate $r'^2 = x'^2+y'^2+z'^2$, finding agreement with Eq. (5.10). Recall that $\sin\alpha$ and $\cos\alpha$ are determined by r_1 and r_2 in Eq. (5.7). This answer is much simpler than Eqs. (5.10)–(5.13), since one is not compounding the complications of curved spaces with the complication of describing a translation in polar coordinates.

For the open universe case, the pseudo-rotation is again described by Eq. (5.22), where w is determined by Eq. (5.31), with $K = -1$. Thus,

$$\boxed{\begin{aligned} x' &= x \\ y' &= y \\ z' &= z\cosh\alpha + \sqrt{1 + x^2 + y^2 + z^2}\,\sinh\alpha\ . \end{aligned}} \tag{5.36}$$