MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.952: Particle Physics of the Early Universe — March 16, 2009
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PROBLEM SET 2 SOLUTIONS

PROBLEM 1: THE MANY COORDINATE SYSTEMS OF DE SITTER SPACE *(20 points)**

(a) The (x, y, z, t) coordinate system is related to the global (V, W, X, Y, Z) system by the equations

$$\begin{aligned} t &= H^{-1} \ln\left[H(W+V)\right] \\ x &= e^{-Ht} X \\ y &= e^{-Ht} Y \\ z &= e^{-Ht} Z \; . \end{aligned} \tag{1.1}$$

The first of these equations implies that

$$W + V = H^{-1} e^{Ht} \; , \tag{1.2}$$

which then implies that the constraint equation

$$X^2 + Y^2 + Z^2 + W^2 - V^2 = X^2 + Y^2 + Z^2 + (W+V)(W-V) = H^{-2} \tag{1.3}$$

can be rewritten as

$$e^{2Ht}(x^2 + y^2 + z^2) + H^{-1} e^{Ht}(W - V) = H^{-2} \; , \tag{1.4}$$

implying

$$W - V = H^{-1} e^{-Ht} - H e^{Ht}(x^2 + y^2 + z^2) \; . \tag{1.5}$$

Taking the sum and difference of Eqs. (1.2) and (1.5),

$$\boxed{\begin{aligned} W &= H^{-1} \cosh Ht - \frac{1}{2} H e^{Ht}(x^2 + y^2 + z^2) \\ V &= H^{-1} \sinh Ht + \frac{1}{2} H e^{Ht}(x^2 + y^2 + z^2) \; . \end{aligned}} \tag{1.6}$$

The remaining equations for the inverse transformation are

$$\boxed{\begin{aligned} X &= e^{Ht} x \\ Y &= e^{Ht} y \\ Z &= e^{Ht} z \; . \end{aligned}} \tag{1.7}$$

The metric

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}X^2 + \mathrm{d}Y^2 + \mathrm{d}Z^2 + \mathrm{d}W^2 - \mathrm{d}V^2 \\ &= \mathrm{d}X^2 + \mathrm{d}Y^2 + \mathrm{d}Z^2 + (\mathrm{d}W + \mathrm{d}V)(\mathrm{d}W - \mathrm{d}V) \end{aligned} \tag{1.8}$$

can then be rewritten by using

$$\begin{aligned} \mathrm{d}X &= e^{Ht}(\mathrm{d}x + Hx\mathrm{d}t) \\ \mathrm{d}Y &= e^{Ht}(\mathrm{d}y + Hy\mathrm{d}t) \\ \mathrm{d}Z &= e^{Ht}(\mathrm{d}z + Hz\mathrm{d}t) \\ \mathrm{d}W + \mathrm{d}V &= e^{Ht}\mathrm{d}t \\ \mathrm{d}W - \mathrm{d}V &= -\left[e^{Ht} H^2 (x^2 + y^2 + z^2) + e^{-Ht}\right] \mathrm{d}t \\ &\qquad - 2e^{Ht} H (x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z) \; . \end{aligned} \tag{1.9}$$

By combining Eqs. (1.8) and (1.9), one finds

$$\boxed{\mathrm{d}s^2 = -\mathrm{d}t^2 + e^{2Ht}(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \; ,} \tag{1.10}$$

which is exactly the flat Robertson-Walker metric that we are seeking. Only half of the full space is covered, because Eq. (1.2) implies that $W + V > 0$.

(b) As the problem explained, for a fixed value of $V = V_0$ the space is a 4-sphere of radius

$$a = \sqrt{V_0^2 + H^{-2}} \; , \tag{1.11}$$

since

$$X^2 + Y^2 + Z^2 + W^2 = a^2 = V_0^2 + H^{-2} \; . \tag{1.12}$$

If we wish to put Robertson–Walker closed universe coordinates on this sphere, with $K = 1$, then r should be a dimensionless coordinate ranging from 0 to 1. This can be arranged by choosing

$$\begin{aligned} X &= a\, r \sin\theta \, \cos\phi \\ Y &= a\, r \sin\theta \, \sin\phi \\ Z &= a\, r \cos\theta \\ W &= a\sqrt{1 - r^2} \; . \end{aligned} \tag{1.13}$$

More compactly, we can define a 3-vector $\mathbf{X}$, with $X^i \equiv (X, Y, Z)$ as i runs from 1 to 3, so then

$$\begin{aligned} \mathbf{X} &= a\, r\, \hat{\boldsymbol{n}}(\theta,\phi) \\ W &= a\sqrt{1-r^2}\ , \end{aligned} \tag{1.14}$$

where

$$\begin{aligned} \hat{\boldsymbol{n}}^1(\theta,\phi) &= \sin\theta\,\cos\phi \\ \hat{\boldsymbol{n}}^2(\theta,\phi) &= \sin\theta\,\sin\phi \\ \hat{\boldsymbol{n}}^3(\theta,\phi) &= \cos\theta\ . \end{aligned} \tag{1.15}$$

We can check that we have the right metric on the 4-sphere by calculating the relevant differentials while holding V fixed. Then

$$\begin{aligned} \mathrm{d}\mathbf{X} &= a\,\hat{\boldsymbol{n}}\,\mathrm{d}r + a\,r\,\mathrm{d}\hat{\boldsymbol{n}} \\ \mathrm{d}W &= -\frac{a}{\sqrt{1-r^2}}\,r\,\mathrm{d}r\ , \end{aligned} \tag{1.16}$$

and

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}\mathbf{X}^2 + \mathrm{d}W^2 \\ &= a^2\,\mathrm{d}r^2 + 2a^2\,r\,\mathrm{d}r\,\hat{\boldsymbol{n}}\cdot\mathrm{d}\hat{\boldsymbol{n}} + a^2\,r^2\,\mathrm{d}\hat{\boldsymbol{n}}^2 + \frac{a^2}{1-r^2}r^2\,\mathrm{d}r^2 \\ &= a^2\left[\frac{\mathrm{d}r^2}{1-r^2} + r^2\mathrm{d}\Omega\right]\ , \end{aligned} \tag{1.17}$$

where

$$\mathrm{d}\Omega = \mathrm{d}\hat{\boldsymbol{n}}^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\ , \tag{1.18}$$

and I used the fact that $\hat{\boldsymbol{n}}\cdot\mathrm{d}\hat{\boldsymbol{n}} = 0$. This is the metric that we wanted. To get the full spacetime metric, we allow V to vary as well, with

$$a = \sqrt{V^2 + H^{-2}}\ , \qquad \mathrm{d}a = \frac{V\,\mathrm{d}V}{a}\ . \tag{1.19}$$

Then

$$\begin{aligned} \mathrm{d}\mathbf{X} &= a\,\hat{\boldsymbol{n}}\,\mathrm{d}r + a\,r\,\mathrm{d}\hat{\boldsymbol{n}} + \frac{r\,\hat{\boldsymbol{n}}}{a}\,V\,\mathrm{d}V \\ \mathrm{d}W &= -\frac{a}{\sqrt{1-r^2}}\,r\,\mathrm{d}r + \frac{\sqrt{1-r^2}}{a}\,V\mathrm{d}V\ , \end{aligned} \tag{1.19}$$

and after some algebra

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}\mathbf{X}^2 + \mathrm{d}W^2 - \mathrm{d}V^2 \\ &= a^2\left[\frac{\mathrm{d}r^2}{1-r^2} + r^2\mathrm{d}\Omega\right] - \frac{\mathrm{d}V^2}{H^2V^2+1}\ . \end{aligned} \tag{1.20}$$

This will match the closed Robertson-Walker form that we are looking for if

$$\mathrm{d}t = \frac{\mathrm{d}V}{\sqrt{H^2V^2+1}}\ , \tag{1.21}$$

which can be integrated to give

$$t = H^{-1}\sinh^{-1}(HV)\ . \tag{1.22}$$

So

$$V = H^{-1}\sinh Ht\ , \tag{1.23}$$

and

$$\boxed{a(t) = \sqrt{V^2 + H^{-2}} = H^{-1}\cosh Ht\ ,} \tag{1.24}$$

where the full metric is then

$$\boxed{\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-r^2} + r^2\mathrm{d}\Omega\right]\ .} \tag{1.25}$$

Finally, we were asked to express r, θ, ϕ, and t in terms of X, Y, Z, W, and V, which we can do by using Eqs. (1.11), (1.13), and (1.22):

$$\boxed{\begin{aligned} r &= \sqrt{\frac{X^2+Y^2+Z^2}{V^2+H^{-2}}} \\ \theta &= \cos^{-1}\left(\frac{Z}{\sqrt{X^2+Y^2+Z^2}}\right) \\ \phi &= \sin^{-1}\left(\frac{Y}{\sqrt{X^2+Y^2}}\right) \\ t &= H^{-1}\sinh^{-1}(HV)\ . \end{aligned}} \tag{1.26}$$

This coordinate system covers the entire spacetime.

As an alternative, one could replace r in the closed universe metric by ξ, where

$$r \equiv \sin\xi\ . \tag{1.27}$$

Then Eqs. (1.13) are replaced by

$$\begin{aligned} X &= a\,\sin\xi\,\sin\theta\,\cos\phi \\ Y &= a\,\sin\xi\,\sin\theta\,\sin\phi \\ Z &= a\,\sin\xi\,\cos\theta \\ W &= a\,\cos\xi\ , \end{aligned} \tag{1.28}$$

and the final metric (Eq. (1.25)) is replaced by

$$\boxed{\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}\xi^2 + \sin^2\xi\,\mathrm{d}\Omega\right]\ .} \tag{1.29}$$

(c) As the problem suggests, we consider the hypersurface $W = W_0$, for which the constraint equation can be written as

$$X^2 + Y^2 + Z^2 - V^2 = -a^2\ , \tag{1.30}$$

where

$$a = \sqrt{W_0^2 - H^{-2}}\ . \tag{1.31}$$

In analogy with Eq. (1.14), we try coordinates

$$\begin{aligned} \mathbf{X} &= a\,r\,\hat{\boldsymbol{n}}(\theta,\phi) \\ V &= a\sqrt{1+r^2}\ , \end{aligned} \tag{1.32}$$

where $\hat{\boldsymbol{n}}(\theta,\phi)$ is again given by Eq. (1.15). As in (b), we can first explore the hypersurface by keeping W (and hence a) fixed. Then

$$\begin{aligned} \mathrm{d}\mathbf{X} &= a\,\hat{\boldsymbol{n}}\,\mathrm{d}r + a\,r\,\mathrm{d}\hat{\boldsymbol{n}} \\ dV &= \frac{a}{\sqrt{1+r^2}}\,r\,\mathrm{d}r\ , \end{aligned} \tag{1.33}$$

and

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}\mathbf{X}^2 - \mathrm{d}V^2 \\ &= a^2\,\mathrm{d}r^2 + 2a^2\,r\,\mathrm{d}r\,\hat{\boldsymbol{n}}\cdot\mathrm{d}\hat{\boldsymbol{n}} + a^2\,r^2\,\mathrm{d}\hat{\boldsymbol{n}}^2 - \frac{a^2}{1+r^2}r^2\,\mathrm{d}r^2 \\ &= a^2\left[\frac{\mathrm{d}r^2}{1+r^2} + r^2\mathrm{d}\Omega\right]\ , \end{aligned} \tag{1.34}$$

which is the spatial part of a Robertson–Walker open universe, as desired. To get the full spacetime metric we allow W to also vary, with

$$a = \sqrt{W^2 - H^{-2}}\ , \qquad \mathrm{d}a = \frac{W\,\mathrm{d}W}{a}\ . \tag{1.35}$$

Then

$$\begin{aligned} \mathrm{d}\mathbf{X} &= a\,\hat{\boldsymbol{n}}\,\mathrm{d}r + a\,r\,\mathrm{d}\hat{\boldsymbol{n}} + \frac{r\,\hat{\boldsymbol{n}}}{a}\,W\,\mathrm{d}W \\ \mathrm{d}V &= \frac{a}{\sqrt{1+r^2}}\,r\,\mathrm{d}r + \frac{\sqrt{1+r^2}}{a}\,W\mathrm{d}W\ , \end{aligned} \tag{1.36}$$

which with some more algebra implies that

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}\mathbf{X}^2 + \mathrm{d}W^2 - \mathrm{d}V^2 \\ &= a^2\left[\frac{\mathrm{d}r^2}{1+r^2} + r^2\mathrm{d}\Omega\right] - \frac{\mathrm{d}W^2}{H^2W^2 - 1}\ . \end{aligned} \tag{1.37}$$

So this time we insist that

$$\mathrm{d}t = \frac{\mathrm{d}W}{\sqrt{H^2W^2 - 1}}\ , \tag{1.38}$$

which integrates to

$$t = H^{-1}\cosh^{-1}(HW)\ , \tag{1.39}$$

so

$$W = H^{-1}\cosh Ht \tag{1.40}$$

and

$$\boxed{a(t) = \sqrt{W^2 - H^{-2}} = H^{-1}\sinh Ht\ .} \tag{1.41}$$

The full metric is then

$$\boxed{\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1+r^2} + r^2\mathrm{d}\Omega\right]\ ,} \tag{1.42}$$

which is exactly the open Robertson–Walker metric that we sought. To express r, θ, ϕ, and t in terms of X, Y, Z, W, and V, use Eqs. (1.31), (1.32), and (1.39), with the result

$$\boxed{\begin{aligned} r &= \sqrt{\frac{X^2+Y^2+Z^2}{W^2-H^{-2}}} \\ \theta &= \cos^{-1}\left(\frac{Z}{\sqrt{X^2+Y^2+Z^2}}\right) \\ \phi &= \sin^{-1}\left(\frac{Y}{\sqrt{X^2+Y^2}}\right) \\ t &= H^{-1}\cosh^{-1}(HW)\ . \end{aligned}} \tag{1.43}$$

This coordinate system does not cover the full de Sitter manifold, since the coordinates have restricted ranges. From Eqs. (1.31) and (1.32), one sees that

$$\boxed{W > H^{-1} \text{ and } V > 0.} \tag{1.44}$$

As in the closed universe case, there is an alternative coordinate system for the open universe in which r is replaced by ξ, where in this case

$$r \equiv \sinh \xi\ . \tag{1.45}$$

Then Eqs. (1.32) are replaced by

$$\begin{aligned} \mathbf{X} &= a \sinh\xi\, \hat{\boldsymbol{n}}(\theta,\phi) \\ V &= a\cosh\xi\ , \end{aligned} \tag{1.46}$$

and the final metric (Eq. (1.42)) is replaced by

$$\boxed{ds^2 = -dt^2 + a^2(t)\left[d\xi^2 + \sinh^2\xi\, d\Omega\right]\ .} \tag{1.47}$$

(d) As stated in the problem, we choose

$$\begin{aligned} V &= \sqrt{H^{-2}-r^2}\sinh Ht \\ W &= \sqrt{H^{-2}-r^2}\cosh Ht\ . \end{aligned} \tag{1.48}$$

Then the de Sitter constraint equation becomes

$$\mathbf{X}^2 = H^{-2} - W^2 + V^2 = r^2\ , \tag{1.49}$$

so the natural parameterization is

$$\mathbf{X} = r\,\hat{\boldsymbol{n}}(\theta,\phi)\ , \tag{1.50}$$

where we again use Eq. (1.15) for $\hat{\boldsymbol{n}}(\theta,\phi)$. Differentiating, we can write

$$\begin{aligned} dV &= H\sqrt{H^{-2}-r^2}\cosh(Ht)\,dt - \frac{r\sinh Ht}{\sqrt{H^{-2}-r^2}}dr \\ dW &= H\sqrt{H^{-2}-r^2}\sinh(Ht)\,dt - \frac{r\cosh Ht}{\sqrt{H^{-2}-r^2}}dr \\ d\mathbf{X} &= dr\,\hat{\boldsymbol{n}} + r\,d\hat{\boldsymbol{n}}\ . \end{aligned} \tag{1.51}$$

It is then straightforward algebra to show that

$$\boxed{\begin{aligned} ds^2 &= d\mathbf{X}^2 + dW^2 - dV^2 \\ &= -(1-H^2r^2)dt^2 + \frac{dr^2}{1-H^2r^2} + r^2 d\Omega\ , \end{aligned}} \tag{1.52}$$

which is exactly the desired metric.

From Eqs. (1.48) we see that

$$\boxed{W \geq 0 \text{ and } |V| \leq |W|\ .} \tag{1.53}$$

Thus the coordinate system covers only one quadrant of the V-W plane. From the final form of the metric, Eq. (1.52), one sees that the metric gives a convenient picture of the universe as seen by a single geodesic observer, at $r = 0$, including all points out to the observer's horizon at $r = H^{-1}$.

PROBLEM 2: THE TRANSITION FROM DECELERATION TO ACCELERATION (Weinberg, Assorted Problem #5, with addition) *(10 points)*†

Suppose that $\Omega_M = 0.25$, $\Omega_\Lambda = 0.75$, and $\Omega_K = \Omega_R = 0$. From the Einstein equations we know one relation involving the acceleration of the scale factor $a(t)$:

$$\frac{\ddot{a}}{a} = -4\pi G(3p+\rho)\ . \tag{2.1}$$

Let t^* be the time since the Big bang ($t = 0$ here) at which the transition to acceleration occured. Then

$$\frac{\ddot{a}(t^*)}{a(t^*)} = 0 \Rightarrow 3p(t^*) + \rho(t^*) = 0 \Rightarrow p(t^*) = -\frac{\rho(t^*)}{3}\,. \tag{2.2}$$

Now matter has $p_M \approx 0$, and vacuum energy has $p_\Lambda = -\rho_\Lambda$. Using this we can write the energy density as a function of scale factor as (Weinberg 1.5.38):

$$\rho(t) = \frac{3H_0^2}{8\pi G}\left[\Omega_\Lambda + \Omega_M\left(\frac{a_0}{a(t)}\right)^3\right]. \tag{2.3}$$

So we can write the result of Eq. (2.2) as

$$-\rho_\Lambda = -\frac{H_0^2}{8\pi G}\left[\Omega_\Lambda + \Omega_M\left(\frac{a_0}{a(t^*)}\right)^3\right]. \tag{2.4}$$

But $\rho_\Lambda = \frac{3H_0^2}{8\pi G}\Omega_\Lambda$ and with $\frac{a_0}{a(t^*)} = 1 + z^*$, where z^* is the value of the redshift that radiation gets by traveling towards us since time t^*, we find by replacing these quantities in Eq. (2.4) that:

$$\Omega_\Lambda = \frac{1}{3}\left[\Omega_\Lambda + \Omega_M(1+z^*)^3\right] \Rightarrow z^* = \left(\frac{2\Omega_\Lambda}{\Omega_M}\right)^{1/3} - 1\,. \tag{2.5}$$

In the end this evaluates to $z^* = 0.817$. To find how long ago the acceleration of the universe started, we just subtract the time we called t^* from the present age of the universe, t_0. From Weinberg equation 1.5.42 the age the universe had when the radiation, arriving to us with redshift z, was released is:

$$t(z) = \frac{1}{H_0}\int_0^{\frac{1}{1+z}} \frac{dx}{x\sqrt{\Omega_\Lambda + \Omega_M x^{-3}}}\,. \tag{2.6}$$

Here I neglected the curvature and radiation contributions. The present time is then $t_0 = t(z=0)$; also $t^* = t(z^*)$, so the time we want to find is:

$$\delta t = t(0) - t(z^*) = \frac{1}{H_0}\int_{\frac{1}{1+z^*}}^{1} \frac{dx}{x\sqrt{\Omega_\Lambda + \Omega_M x^{-3}}}\,. \tag{2.7}$$

If we then further use that $\Omega_\Lambda + \Omega_K + \Omega_R + \Omega_M = 1$ and neglect the radiation and curvature contributions, we can do this integral analytically. Inserting $\Omega_M = 1 - \Omega_\Lambda$, $t(0) = 13.7\,\text{Gyr}$ we find:

$$\begin{aligned}\delta t &= \frac{1}{H_0}\int_{\frac{1}{1+z^*}}^{1} \frac{dx}{x\sqrt{\Omega_\Lambda + (1-\Omega_\Lambda)x^{-3}}} \\ &= \frac{2}{3H_0\sqrt{\Omega_\Lambda}}\log\left[\sqrt{\Omega_\Lambda}\,x^{3/2} + \sqrt{\Omega_\Lambda(x^3-1)+1}\right]\Big|_{x=\frac{1}{1+z^*}}^{x=1} \\ &\Rightarrow \delta t \approx (1/H_0)(0.51)\,.\end{aligned} \tag{2.8}$$

Here the value $1/H_0$ is determined from the given age of 13.7 Gyr:

$$\begin{aligned}t_0 &= \frac{1}{H_0}\int_0^1 \frac{dx}{x\sqrt{\Omega_\Lambda + (1-\Omega_\Lambda)x^{-3}}} \\ &= \frac{2}{3H_0\sqrt{\Omega_\Lambda}}\,\text{ArcSinh}\left(\sqrt{\frac{\Omega_\Lambda}{1-\Omega_\Lambda}}\right) \\ &\Rightarrow t_0 \approx (1/H_0)(1.014) \Rightarrow 1/H_0 \approx t_0/1.014 \approx 13.5\,\text{Gyr}\end{aligned} \tag{2.9}$$

Thus $\delta t \approx (1/H_0)(0.51) \approx 6.9\,\text{Gyr}$.

PROBLEM 3: THE VIRIAL THEOREM WITH A HYPOTHETICAL FORCE LAW (Weinberg, Assorted Problem #5) *(10 points)*†

Consider a cluster of point masses m_n, with coordinates X_n^i, $i = 1, 2, 3$, with respect to the center of mass of the system. Let's start out with the equation in Weinberg 1.9.3:

$$-\sum_{n,i} X_n^i \frac{\partial V_{\text{cluster}}}{\partial X_n^i} = \frac{1}{2}\frac{d^2}{dt^2}\left(\sum_n m_n \boldsymbol{X}_n^2\right) - 2T. \tag{3.1}$$

In the last equation, T is the kinetic energy of the system due to motion about its center of mass. V_{cluster} is the total potential energy of the cluster. The assumption of virialization makes the total time derivative term on the right hand side vanish so we are left with

$$\sum_{n,i} X_n^i \frac{\partial V_{\text{cluster}}}{\partial X_n^i} = 2T. \tag{3.2}$$

For a two body interaction between bodies l and p with a potential of the form $V(r_{lp}) = -G\,c_{lp}/|\boldsymbol{r}_l - \boldsymbol{r}_p|^n$ with $c_{lp} = m_l\,m_p$ the partial derivative of V_{cluster} becomes:

$$\begin{aligned}V_{\text{cluster}} &= -\frac{1}{2}\sum_{m\neq l} \frac{G\,c_{ml}}{|\boldsymbol{r}_m - \boldsymbol{r}_l|^n} \\ \Longrightarrow \quad \frac{\partial V_{\text{cluster}}}{\partial X_q^i} &= -\frac{1}{2}\sum_{m\neq l} \frac{-n\,G\,c_{ml}}{|\boldsymbol{r}_m - \boldsymbol{r}_l|^{n+1}} \frac{\partial|\boldsymbol{r}_m - \boldsymbol{r}_l|}{\partial X_q^i} \\ &= \frac{n}{2}\sum_{m\neq l} \frac{G\,c_{ml}}{|\boldsymbol{r}_m - \boldsymbol{r}_l|^{n+2}}\left(\boldsymbol{r}_l - \boldsymbol{r}_m\right)\cdot\left(\delta_m^q - \delta_l^q\right)\hat{\boldsymbol{e}}_i\,,\end{aligned} \tag{3.3}$$

where $\hat{\mathbf{e}}_i$ is the unit vector along the ith direction. Multiplying by X^i_q and summing over i and q we find:

$$\begin{aligned}\sum_{q,i} X^i_q \frac{\partial V_{\rm cluster}}{\partial X^i_q} &= \sum_{q,i,m\neq l} X^i_q \left[\frac{n}{2}\frac{G\,c_{ml}}{|\boldsymbol{r}_m-\boldsymbol{r}_l|^{n+2}}\Big(\boldsymbol{r}_l-\boldsymbol{r}_m\Big)\cdot\Big(\delta^q_m-\delta^q_l\Big)\hat{\mathbf{e}}_i\right] \\ &= \frac{n}{2}\sum_{m\neq l}\frac{G\,c_{ml}}{|\boldsymbol{r}_m-\boldsymbol{r}_l|^{n+2}}\Big(\boldsymbol{r}_m-\boldsymbol{r}_l\Big)\cdot\Big(\boldsymbol{r}_m-\boldsymbol{r}_l\Big) \\ &= n\left[\frac{1}{2}\sum_{m\neq l}\frac{G\,c_{ml}}{|\boldsymbol{r}_m-\boldsymbol{r}_l|^{n}}\right] \\ &= -n\,V_{\rm cluster}.\end{aligned} \tag{3.4}$$

Here we used $\boldsymbol{r}_l = \sum_i X^i_l\,\hat{\mathbf{e}}_i$. So the virial theorem then takes the form

$$2\,T = -n\,V_{\rm cluster}. \tag{3.5}$$

One can get write the kinetic energy in terms of the mass-averaged square velocity relative to the center of mass, $\langle v^2\rangle$ as $T = (1/2)\,M\,\langle v^2\rangle$, where M is the total mass of the cluster. A similar thing can be done with $V_{\rm cluster}$ by considering the mass-averaged value of $1/r^n$, where r is the separation between any two masses in the cluster. It becomes $V_{\rm cluster} = -(1/2)\,GM^2\langle(1/r^n)\rangle$. Thus using the virial theorem result we can find the total mass M of the cluster:

$$\begin{aligned} 2\,T &= -n\,V_{\rm cluster} \\ \Longrightarrow\quad 2\left(\frac{1}{2}M\,\langle v^2\rangle\right) &= -n\left(-\frac{1}{2}GM^2\left\langle\frac{1}{r^n}\right\rangle\right) \\ \Longrightarrow\quad M &= \frac{2\,\langle v^2\rangle}{n\,G\,\langle\frac{1}{r^n}\rangle}.\end{aligned} \tag{3.6}$$

The values of $\langle v^2\rangle$ can be obtained from the velocity dispersion arising from Doppler shifts in the spectra coming from the visible galaxies — assuming that in statistical equilibrium the visible masses are representative sample of the virialized cluster. For $\langle(1/r^n)\rangle$, we can estimate it for clusters with $z \ll 1$. For small z, the angular diameter distance of a cluster $d_A \approx z/H_0$ (from Weinberg 1.4.9 and 1.4.11). Since the transverse proper distance is related to the angular separation θ and d_A as $d = \theta\,d_A$ you get $d \approx \theta\,z/H_0$. Thus $M \propto 1/H_0^n$. Even if we go to higher z, at which the dependence of d_A on redshift ceases to be linear, we still expect $M \propto 1/H_0^n$.

PROBLEM 4: TIME OF EMISSION OF LIGHT FROM A VERY DISTANT GALAXY *(10 points)**

The age of the universe at the time of emission of light that reaches us at redshift z is given by Weinberg's Eq. (1.5.42):

$$t(z) = \frac{1}{H_0}\int_0^{1/(1+z)} \frac{{\rm d}x}{x\sqrt{\Omega_\Lambda + \Omega_K x^{-2} + \Omega_M x^{-3} + \Omega_R x^{-4}}}\ , \tag{4.1}$$

where $\Omega_K = 1 - \Omega_\Lambda - \Omega_M - \Omega_R$. For the special case of a matter-dominated flat universe, with $\Omega_M = 1$, $\Omega_\Lambda = \Omega_K = \Omega_R = 0$, the integral is easily carried out, giving

$$t(z) = \frac{2}{3H_0}\frac{1}{(1+z)^{3/2}}\ . \tag{4.2}$$

The age of such a universe is given by

$$t_0 = t(0) = \frac{2}{3H_0}\ , \tag{4.3}$$

so

$$t(z) = \frac{t_0}{(1+z)^{3/2}}\ . \tag{4.4}$$

For $t_0 = 13.7$ Gyr and $z = 6.96$, this gives

$$\boxed{\left. t(6.96)\right|_{\substack{\text{matter}\\ \text{only}}} = \frac{13.7\ \text{Gyr}}{(1+6.96)^{3/2}} = 0.610\ \text{Gyr}.} \tag{4.5}$$

That is, $t(z)$ is 610 million years.

For the realistic model based on the WMAP 5-year recommended values, as described in the problem, the integral has to be done numerically, using the conversion

$$\frac{1}{H_0} = 9.778\ h^{-1}\ \text{Gyr}\ , \tag{4.6}$$

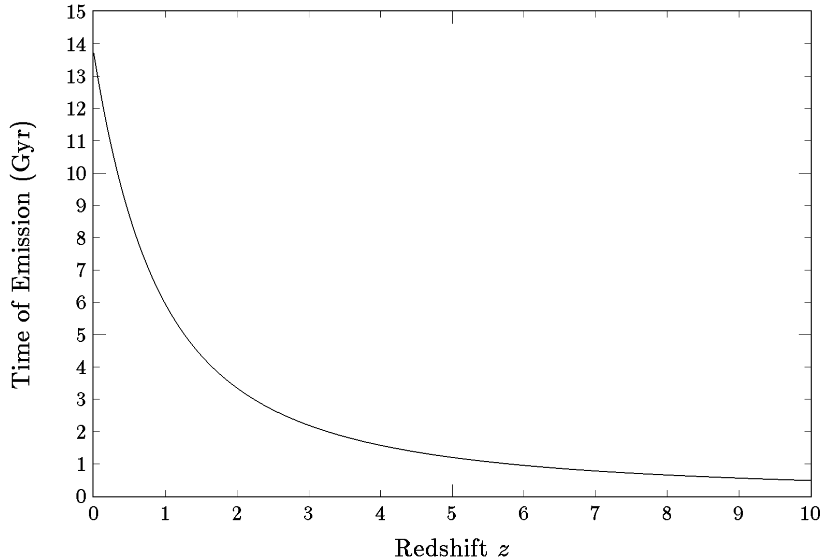
where

$$H_0 = 100\,h\,\text{km s}^{-1}\text{Mpc}^{-1}\ , \tag{4.7}$$

with $\Omega_M = \Omega_b + \Omega_{\rm dm} = 0.0456 + 0.228 = 0.2736$. The integrations give an age of $t_0 = 13.71$ Gyr and a time of emission for the $z = 6.96$ galaxy given by

$$\boxed{\left. t(6.96)\right|_{\substack{\text{WMAP5}\\ \text{parameters}}} = 0.784\ \text{Gyr}.} \tag{4.8}$$

You were not asked to draw a graph, but numerical integration using the WMAP 5-year recommended parameters leads to the following:



A useful special case is that of a flat universe with matter and vacuum energy, so $\Omega_R = \Omega_K = 0$, with $\Omega_\Lambda = 1 - \Omega_M$. In that case the integral can also be done analytically, with the result

$$t(z)\Bigg|_{\substack{\text{matter/vacuum}\\ \text{only}}} = \frac{2}{3H_0\sqrt{\Omega_\Lambda}} \operatorname{arcsinh}\left[\frac{\sqrt{\Omega_\Lambda}}{\sqrt{\Omega_M}\,(1+z)^{3/2}}\right] . \qquad (4.9)$$

Using the WMAP5 values for Ω_M and H_0, this approximation gives an age $t_0 =$ 13.72 Gyr and $t(6.96) = 0.786$ Gyr, which are both very close to the values found above for the full numerical integral.

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*Solution written by Alan Guth.

†Solution written by Carlos Santana.