MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.952: Particle Physics of the Early Universe April 7, 2009
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PROBLEM SET 3 SOLUTIONS

PROBLEM 1: DISTANCE TO A GALAXY AT $z = 6.96$ *(10 points)**

Summarizing the key formulas, the Robertson–Walker metric can be written as

$$ds^2 = -dt^2 + a^2(t)\left\{d\xi^2 + S_K^2(\xi)\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right\} , \qquad (1.1)$$

where ξ is related to the frequently used coordinate r by

$$r = S_K(\xi) \equiv \begin{cases} \sin\xi & \text{if } K = 1 \\ \xi & \text{if } K = 0 \\ \sinh\xi & \text{if } K = -1 \ . \end{cases} \qquad (1.2)$$

If the redshift of the light that we now receive from a distant object is z, then its radial coordinate ξ is given by

$$\xi(z) = \frac{1}{a(t_0)H_0}\int_{1/(1+z)}^{1} \frac{dx}{x^2\sqrt{\Omega_\Lambda + \Omega_M x^{-3} + \Omega_R x^{-4} + \Omega_K x^{-2}}} , \qquad (1.3)$$

where

$$\Omega_K = 1 - \Omega_{\text{tot}} = 1 - \Omega_\Lambda - \Omega_M - \Omega_R . \qquad (1.4)$$

For $K = \pm 1$, the coefficient in Eq. (1.3) can be rewritten, giving

$$\xi(z) = \sqrt{\frac{\Omega_K}{-K}}\int_{1/(1+z)}^{1} \frac{dx}{x^2\sqrt{\Omega_\Lambda + \Omega_M x^{-3} + \Omega_R x^{-4} + \Omega_K x^{-2}}} . \qquad (1.5)$$

In terms of these quantities, the three definitions of distance — proper distance $d(z)$, luminosity distance $d_L(z)$, and angular diameter distance $d_A(z)$ — are given by

$$\begin{aligned} d(z) &= a(t_0)\xi(z) \\ d_L(z) &= (1+z)\, a(t_0)\, S_K\big(\xi(z)\big) \\ d_A(z) &= (1+z)^{-1}\, a(t_0)\, S_K\big(\xi(z)\big) , \end{aligned} \qquad (1.6)$$

where for $K = \pm 1$, $a(t_0)$ can be rewritten as

$$a(t_0) = \frac{1}{H_0}\sqrt{\frac{-K}{\Omega_K}} . \qquad (1.7)$$

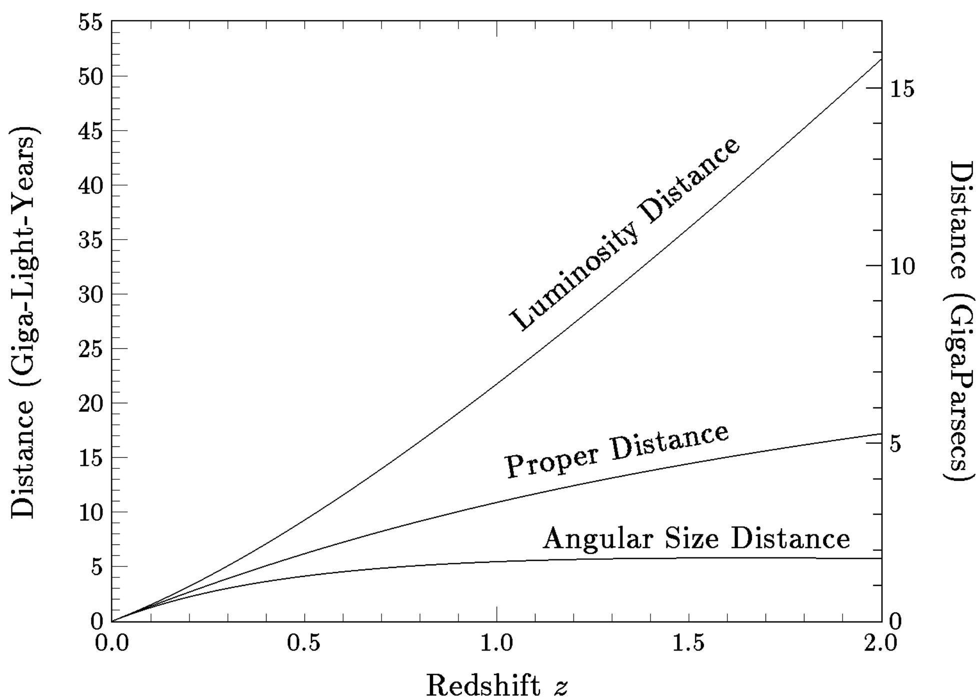
To continue, we use the WMAP 5-year recommended values: $H_0 = 70.5\,\text{km}\cdot\text{s}^{-1}\cdot\text{Mpc}^{-1}$, $\Omega_M = \Omega_b + \Omega_{\text{dm}} = 0.2736$, $\Omega_\Lambda = 0.726$, and $\Omega_R = 8.4\times10^{-5}$. If these values are taken literally, then $\Omega_K = 3.2 \times 10^{-4}$, but this number is consistent with zero. One can proceed numerically either by using the $K = -1$ formulas or the $K = 0$ formulas, and to three significant figures the results will agree:

$$\boxed{\begin{aligned} d(6.96) &= 28.8 \text{ GLYr} = 8.83 \text{ Gpc} \\ d_L(6.96) &= 229 \text{ GLYr} = 70.3 \text{ Gpc} \\ d_A(6.96) &= 3.62 \text{ GLYr} = 1.11 \text{ Gpc}, \end{aligned}} \tag{1.8}$$

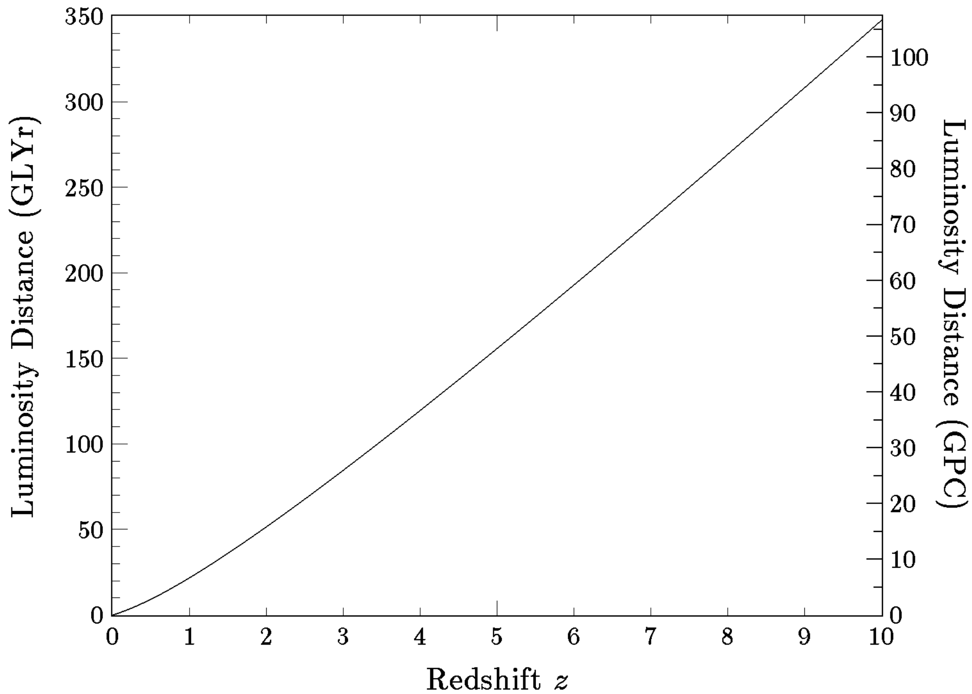
where GLYr = giga-light-year, and Gpc = gigaparsec. Note that for a flat universe, all three measures of distance are simply related:

$$d_L(z) = (1+z)\, d(z), \qquad d_A(z) = (1+z)^{-1}\, d(z) \ . \tag{1.9}$$

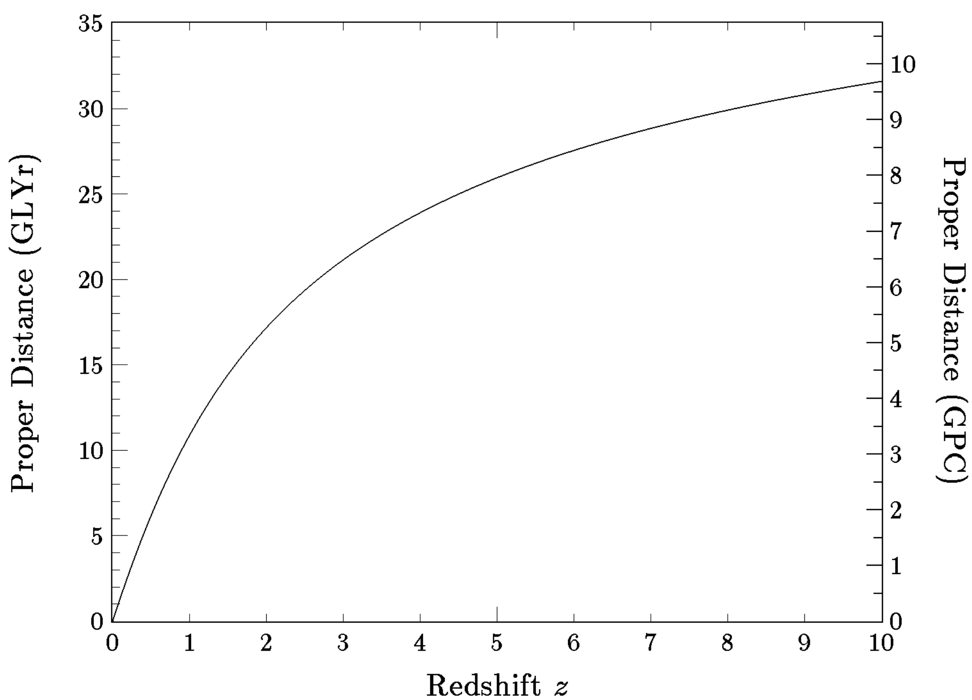
You were not asked to do so, but it is interesting to graph the three measures of distance vs. z, using the WMAP 5-year recommended values. For small z one can put them on the same graph:



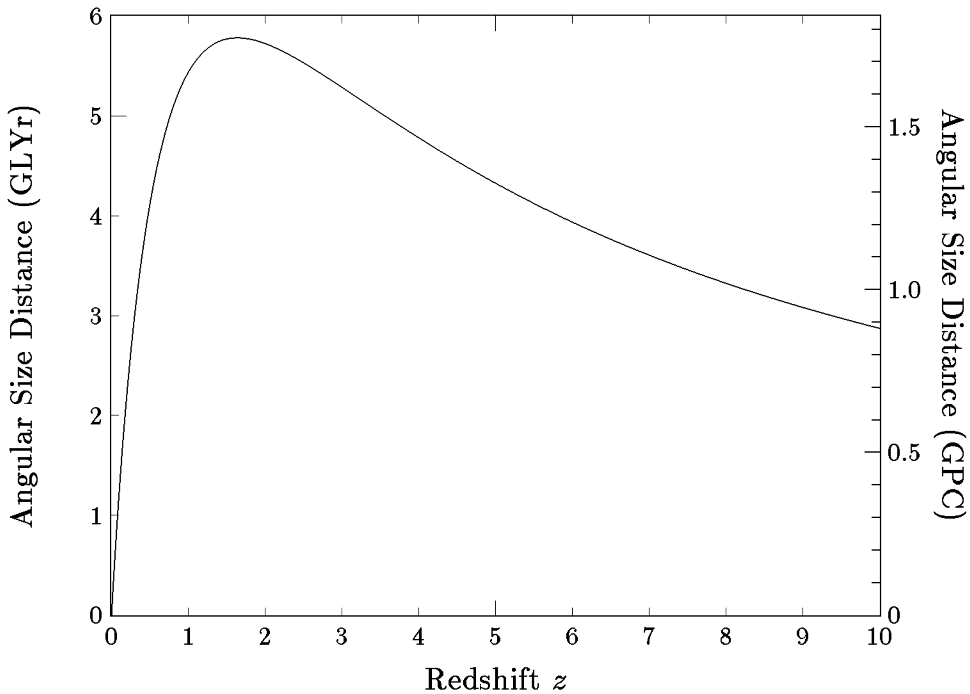
For larger z they are so different from each other that it is clearer to graph them separately:



Luminosity Distance (GLYr)
Luminosity Distance (GPC)
Redshift z
0
50
100
150
200
250
300
350
0
10
20
30
40
50
60
70
80
90
100
0
1
2
3
4
5
6
7
8
9
10



Proper Distance (GLYr)
Proper Distance (GPC)
Redshift z
0
5
10
15
20
25
30
35
0
1
2
3
4
5
6
7
8
9
10
0
1
2
3
4
5
6
7
8
9
10



PROBLEM 2: VELOCITY OF DISTANT GALAXIES *(10 points)**

Since the proper distance can be expressed as

$$d(t) = a(t)\,\xi \ , \tag{2.1}$$

where ξ is the (time-independent) coordinate distance, one has the standard Hubble velocity law,

$$v = \frac{\mathrm{d}}{\mathrm{d}t} d(t) = \frac{\mathrm{d}a}{\mathrm{d}t}\,\xi = \left(\frac{1}{a(t)}\,\frac{\mathrm{d}a}{\mathrm{d}t}\right)\big(a(t)\,\xi\big) = H\,d(t) \ . \tag{2.2}$$

For the Einstein–de Sitter model,

$$\begin{aligned} d(z) &= \frac{1}{H_0}\int_{1/(1+z)}^{1} \frac{\mathrm{d}x}{x^2\sqrt{\Omega_\Lambda + \Omega_M x^{-3} + \Omega_R x^{-4} + \Omega_K x^{-2}}} \\ &= \frac{1}{H_0}\int_{1/(1+z)}^{1} \frac{\mathrm{d}x}{x^{1/2}} \\ &= \frac{2}{H_0}\left[1 - \frac{1}{\sqrt{1+z}}\right] \ , \end{aligned} \tag{2.3}$$

and therefore

$$\boxed{v = 2\left[1 - \frac{1}{\sqrt{1+z}}\right] .} \tag{2.4}$$

Alternatively, one can derive Eq. (2.4) by starting with

$$a(t) = \beta t^{2/3} \tag{2.5}$$

for some constant β. Then a light pulse emitted at time t_e and received today ($t = t_0$) has traveled a proper distance

$$d = a(t_0)\int_{t_e}^{t_0} \frac{\mathrm{d}t}{a(t)} = 3t_0\left[1 - \left(\frac{t_e}{t_0}\right)^{1/3}\right] . \tag{2.6}$$

By using

$$H_0 = \frac{\dot{a}(t_0)}{a(t_0)} = \frac{2}{3t_0} \tag{2.7}$$

and

$$1 + z = \frac{a(t_0)}{a(t_e)} = \left(\frac{t_0}{t_e}\right)^{2/3} , \tag{2.8}$$

one sees that Eq. (2.6) reproduces Eq. (2.3) and therefore (2.4).

From Eq. (2.4), $v = 1$ occurs at redshift

$$\begin{gathered} 1 = 2\left[1 - \frac{1}{\sqrt{1+z}}\right] \quad \Longrightarrow \quad \frac{1}{\sqrt{1+z}} = \frac{1}{2} \\ \Longrightarrow \quad \sqrt{1+z} = 2 \quad \Longrightarrow \quad \boxed{z = 3 .} \end{gathered} \tag{2.9}$$

In general one has

$$v = H_0\, d(z) = \int_{1/(1+z)}^{1} \frac{\mathrm{d}x}{x^2\sqrt{\Omega_\Lambda + \Omega_M x^{-3} + \Omega_R x^{-4} + \Omega_K x^{-2}}} , \tag{2.10}$$

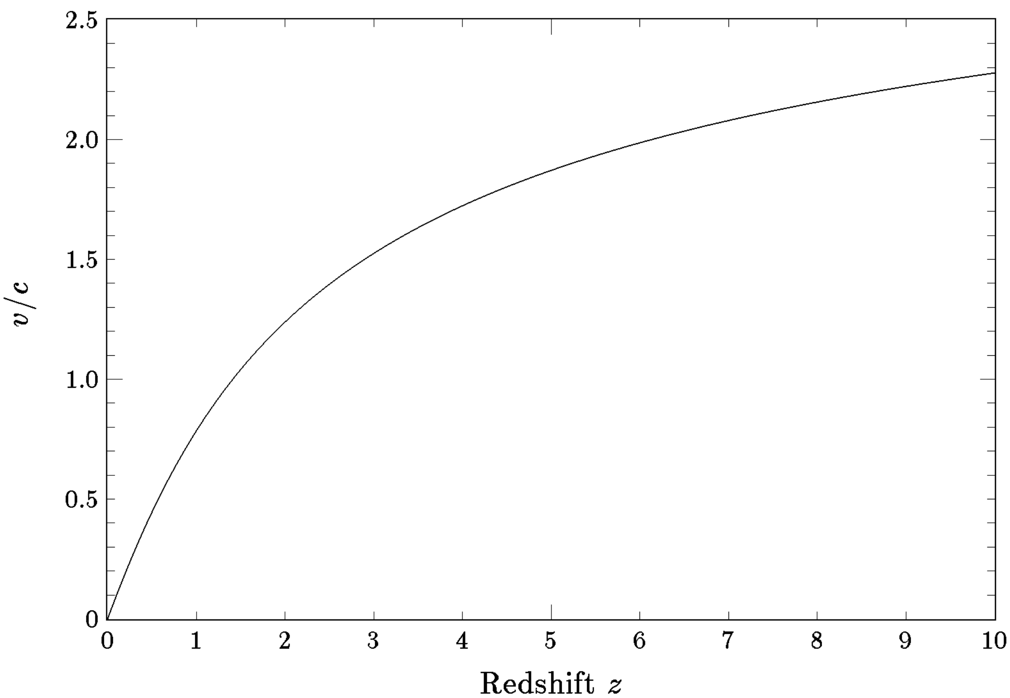
which for the WMAP 5-year recommended parameters and $z = 6.96$ integrates numerically to give

$$\boxed{\frac{v}{c} = 2.076 .} \tag{2.11}$$

Setting $v = 1$ in Eq. (2.10), one can solve it numerically to find that the speed of light is achieved at

$$\boxed{z = 1.412\ .} \tag{2.12}$$

Again you were not asked to draw a graph for v vs. z, but it is interesting to see such a graph. Using the WMAP 5-year parameters, one finds



PROBLEM 3: ANGULAR DIAMETER DISTANCE *(10 points)*†

For this problem we start with the formula for angular diameter distance given in Weinberg (1.10.15):

$$d_A(z) = \frac{1}{(1+z)H_0\,\Omega_K^{1/2}} \sinh\left[\Omega_K^{1/2} \int_{1/(1+z)}^{1} \frac{dx}{x^2\sqrt{\Omega_\Lambda + \Omega_K x^{-2} + \Omega_M x^{-3} + \Omega_R x^{-4}}}\right]. \tag{3.1}$$

In the Einstein-de Sitter model, $\Omega_K = \Omega_\Lambda = \Omega_R = 0$ and $\Omega_M = 1$. For the expression above, we use the fact that for small Ω_K (and representing the integral above by $f(\Omega_K, z)$):

$$\frac{\sinh\left(\Omega_K^{1/2}\, f(\Omega_K, z)\right)}{\Omega_K^{1/2}} = f(0, z) + \ldots\ . \tag{3.2}$$

Using this information, we can obtain the analyic expression for $d_A(z)$,

$$\begin{aligned} d_A(z) &= \frac{1}{(1+z)H_0} f(0,z) = \frac{1}{(1+z)H_0} \int_{1/(1+z)}^{1} \frac{dx}{x^2\sqrt{x^{-3}}} \\ &= \frac{1}{(1+z)H_0} \int_{1/(1+z)}^{1} \frac{dx}{x^{1/2}} \\ &= \frac{2}{(1+z)H_0} \left[1 - \frac{1}{\sqrt{1+z}}\right] . \end{aligned} \tag{3.3}$$

Computing the derivative with respect to z and finding the extremum, we find that

$$\begin{aligned} d'_A(z) &= \frac{1}{H_0} \left[\frac{3}{(1+z)^{5/2}} - \frac{2}{(1+z)^2}\right] , \\ \text{so } \; d'_A(z^*) &= 0 \quad \Longrightarrow \quad z^* = 5/4. \end{aligned} \tag{3.4}$$

Using the z^* that we just found and putting it in our expression for $d_A(z)$ gives $d_A(z^*) = \frac{8}{27H_0}$.

Now for the more realistic scenario, we use the WMAP 5-year recommended values $H_0 = 70.5\,\text{km}\cdot\text{s}^{-1}\cdot\text{Mpc}^{-1}$, $\Omega_M = \Omega_b + \Omega_{\text{dm}} = 0.2736$, $\Omega_\Lambda = 0.726$, $\Omega_R = 8.4\times 10^{-5}$. Also curvature is negligible, so $\Omega_K = 0$. So our expression for angular diameter distance becomes

$$d_A(z) = \frac{1}{(1+z)H_0} \int_{1/(1+z)}^{1} \frac{dx}{x^2\sqrt{\Omega_\Lambda + \Omega_M x^{-3} + \Omega_R x^{-4}}} . \tag{3.5}$$

To find the maximum of this function, it has to be done numerically. Using the Mathematica function FindRoot we can get the value of z that maximizes $d_A(z)$ as $z^* \approx 1.639$. Putting this value back into $d_A(z)$ and evaluating the integral numerically we obtain $d_A(z^*) \approx 0.417/H_0 \approx 5.78$ GLYr ≈ 1.773 Gpc.

PROBLEM 4: SAHA EQUATION *(10 points)*†

Let's write for reference the Saha equation as written in Weinberg (2.3.6) and (2.3.7),

$$X(1+SX) = 1, \tag{4.1}$$

where $X = n_p/(n_p + n_{1s})$ is the fractional hydrogen ionization, and

$$S = 0.76\, n_B \left(\frac{m_e kT}{2\pi\hbar^2}\right)^{-3/2} \exp\left(B_1/kT\right) . \tag{4.2}$$

Notice that in this problem, we are told to find the temperature T at which we would obtain $X = 1/2$ under different values of the parameters. This will involve solving the Saha equation numerically. In order to facilitate this, it is useful to use the espression for S given in Weinberg (2.3.8):

$$S = 1.747 \times 10^{-22}\, e^{157894/T}\, T^{3/2}\, \Omega_B h^2 \ . \tag{4.3}$$

Here T is the temperature in degrees Kelvin and h is the Hubble constant in units of $100\,\text{km} \cdot \text{s}^{-1} \cdot \text{Mpc}^{-1}$.

Now let's proceed to the numerical solution. Again, the Mathematica function FindRoot was used.

1. For the case of the original parameters given in Problem 1, the temperature for $X = 1/2$ is $T_{1/2} = 3738\,\text{K}$.
2. For the case of having a mass of the electron m_e that is twice as large as the real value — assuming the binding energy B_1 stays constant — changes S by a factor of $2^{-3/2}$. The result is that $T_{1/2} = 3645\,\text{K}$.
3. For the case of having a mass of the electron m_e that is half as large as the real value — assuming the binding energy B_1 stays constant — changes S by a factor of $(1/2)^{-3/2}$. The result is that $T_{1/2} = 3836\,\text{K}$.
4. For the case of having a binding energy B_1 that is twice as large as the real value, we obtain $T_{1/2} = 7672\,\text{K}$. Notice the jump in the temperature with respect to the previous two cases. This sensitivity arises due to the exponential dependence on the binding energy.
5. For the case of having a binding energy B_1 that is half as large as the real value, we obtain $T_{1/2} = 1822\,\text{K}$. Notice how much the universe must cool prior to reaching $X = 1/2$!
6. Setting Ω_B to be 10 times larger than the WMAP 5-year values, we find $T_{1/2} = 3962\,\text{K}$.
7. Setting Ω_B to be 10 times smaller than the WMAP 5-year values, we find $T_{1/2} = 3538\,\text{K}$.

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*Solution written by Alan Guth.

†Solution written by Carlos Santana.