MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

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PROBLEM SET 4 SOLUTIONS

PROBLEM 1: CANONICAL FORMULATION OF GEODESIC MOTION IN GENERAL RELATIVITY *(15 points)*†

(a) For this problem we start with the formula for the proper time along a trajectory $x^\mu(s)$ parametrized by s:

$$\tau = \int_{s_1}^{s_2} \sqrt{-g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}}\, ds. \tag{1}$$

Taking the variation of τ with respect to x^μ,

$$\begin{aligned}\delta\tau &= \int_{s_1}^{s_2} \delta\sqrt{-g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}}\, ds \\ &= \int_{s_1}^{s_2} \frac{1}{2\sqrt{-g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}}}\left(-\delta g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - 2g_{\mu\nu}\frac{dx^\mu}{ds}\frac{d\delta x^\nu}{ds}\right) ds \\ &= -\frac{1}{2}\int_{s_1}^{s_2}\left(\frac{1}{\sqrt{A}}\left[\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\delta x^\lambda\right] + \frac{2}{\sqrt{A}}g_{\mu\nu}\frac{dx^\mu}{ds}\frac{d\delta x^\nu}{ds}\right) ds.\end{aligned} \tag{2}$$

The expression $A = -g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}$. Upon integrating by parts in (2) using the condition that $\delta x^\mu(s_1) = \delta x^\mu(s_2) = 0$, and extremizing by setting the variation of τ to zero we find:

$$\begin{aligned}\delta\tau &= -\frac{1}{2}\int_{s_1}^{s_2}\frac{1}{\sqrt{A}}\left[\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - 2\frac{d}{ds}\left(g_{\mu\lambda}\frac{dx^\mu}{ds}\right)\right]\delta x^\lambda\, ds \\ \Longrightarrow\quad & \delta\tau = 0\ \forall\, \delta x^\lambda \\ \Longrightarrow\quad & \frac{d}{ds}\left(\frac{1}{\sqrt{A}}g_{\mu\lambda}\frac{dx^\mu}{ds}\right) = \frac{1}{2\sqrt{A}}\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds},\end{aligned} \tag{3}$$

which is the desired expression.

(b) Now we choose the proper time τ as the parameter, so $s = \tau$. This choice sets $A = 1$. Using this fact in the result obtained in part (a), and expanding the derivative in (3) we find:

$$\begin{aligned}\frac{d}{d\tau}\left(g_{\mu\lambda}\frac{dx^\mu}{d\tau}\right) &= \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \\ \Longrightarrow\quad \frac{dg_{\mu\lambda}}{d\tau}\frac{dx^\mu}{d\tau} + g_{\lambda\mu}\frac{d^2x^\mu}{d\tau^2} &= \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \\ \Longrightarrow\quad \frac{\partial g_{\lambda\mu}}{\partial x^\alpha}\frac{dx^\alpha}{d\tau}\frac{dx^\mu}{d\tau} + g_{\lambda\mu}\frac{d^2x^\mu}{d\tau^2} &= \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\end{aligned} \tag{4}$$

Solving for the second derivative of x^μ with respect to τ and manipulating indices,

$$\begin{aligned}g_{\lambda\mu}\frac{d^2x^\mu}{d\tau^2} &= -\frac{\partial g_{\lambda\mu}}{\partial x^\alpha}\frac{dx^\alpha}{d\tau}\frac{dx^\mu}{d\tau} + \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \\ \Longrightarrow\quad g_{\lambda\mu}\frac{d^2x^\mu}{d\tau^2} &= -\left[\frac{1}{2}\left(\frac{\partial g_{\lambda\mu}}{\partial x^\alpha} + \frac{\partial g_{\lambda\alpha}}{\partial x^\mu} - \frac{\partial g_{\mu\alpha}}{\partial x^\lambda}\right)\right]\frac{dx^\mu}{d\tau}\frac{dx^\alpha}{d\tau} \\ \Longrightarrow\quad \frac{d^2x^\beta}{d\tau^2} &= -\left[\frac{1}{2}g^{\beta\lambda}\left(\frac{\partial g_{\lambda\mu}}{\partial x^\alpha} + \frac{\partial g_{\lambda\alpha}}{\partial x^\mu} - \frac{\partial g_{\mu\alpha}}{\partial x^\lambda}\right)\right]\frac{dx^\mu}{d\tau}\frac{dx^\alpha}{d\tau} \\ \Longrightarrow\quad \frac{d^2x^\beta}{d\tau^2} &= -\Gamma^\beta_{\mu\alpha}\frac{dx^\mu}{d\tau}\frac{dx^\alpha}{d\tau}\end{aligned} \tag{5}$$

Which is the conventional presentation of the geodesic equation.

(c) Starting out with the Lagrangian

$$L = -m\sqrt{-g_{\mu\nu}(x^i,t)\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}} \tag{6}$$

$$= -m\sqrt{A} \tag{7}$$

$$= -m\sqrt{-g_{00} - 2g_{0i}\frac{dx^i}{dt} - g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}} \tag{8}$$

and using the notation $\dot{x}^i = \frac{dx^i}{dt}$ we can find the canonical momenta:

$$p_i = \frac{\partial L}{\partial \dot{x}^i} \tag{9}$$

$$\begin{aligned}&= \frac{m}{\sqrt{A}}\left[g_{i0} + g_{ij}\frac{dx^j}{dt}\right] \\ &= \frac{m}{\sqrt{A}}\left[g_{i0}\frac{dx^0}{dt} + g_{ij}\frac{dx^j}{dt}\right].\end{aligned} \tag{10}$$

Now $d\tau = \sqrt{A}\,dt$, so upon using the chain rule with $\frac{d\tau}{dt} = \sqrt{A}$ we quickly get:

$$\begin{aligned}p_i &= m\left[g_{i0}\frac{dx^0}{d\tau} + g_{ij}\frac{dx^j}{d\tau}\right] \\ \Longrightarrow\quad p_i &= mg_{i\nu}\frac{dx^\nu}{d\tau}\end{aligned} \tag{11}$$

which are the spatial components of $p_\mu = mg_{\mu\nu}\frac{dx^\nu}{d\tau}$.

(d) We can construct the Hamiltonian in the usual way:

$$H(x^i, p_i, t) = p_i \frac{dx^i}{dt} - L \tag{12}$$

$$= \sqrt{A}\left(p_i \frac{dx^i}{d\tau} + m\right), \tag{13}$$

where the chain rule and Eq. (7) were used. To relate this to p_0, recall that p_0 is defined in terms of the canonical variables by

$$p^2 = g^{\mu\nu}(x^i, t) p_\mu p_\nu = -m^2 \ . \tag{14}$$

Given Eq. (11) for p_i, it is easily seen that Eq. (14) is satisfied if we set

$$p_0 = m g_{0\nu} \frac{dx^\nu}{d\tau} \ , \tag{15}$$

so

$$p^\mu = m \frac{dx^\mu}{d\tau} \tag{16}$$

for all μ, as usual. Eq. (13) can then be written

$$\begin{aligned} H(x^i, p_i, t) &= \frac{1}{m}\frac{d\tau}{dt}\left(p_i\, p^i + m^2\right) = -\frac{1}{m}\frac{d\tau}{dt}\left(p_0\, p^0\right) \\ &= -\frac{1}{m}\frac{d\tau}{dt}\left(p_0\, m \frac{dt}{d\tau}\right) = -p_0 \ , \end{aligned} \tag{17}$$

where we have used Eqs. (14) and (16).

(e) We can determine the form of Hamilton's equations by differentiating the defining equation for p_0, Eq. (14). Starting with the equation for $\dot{x}^i = \partial H / \partial p_i$,

$$\begin{aligned} \frac{\partial}{\partial p_i}\left[g^{\mu\nu} p_\mu p_\nu = -m^2\right] \quad &\Longrightarrow \quad 2 g^{i\nu} p_\nu + 2 g^{0\nu} \frac{\partial p_0}{\partial p_i} p_\nu = 0 \\ &\Longrightarrow \quad p^i + p^0 \frac{\partial p_0}{\partial p_i} = 0 \\ &\Longrightarrow \quad \dot{x}^i = \frac{\partial H}{\partial p_i} = -\frac{\partial p_0}{\partial p_i} = \frac{p^i}{p^0} \ . \end{aligned} \tag{18}$$

Similarly for $\dot{p}_i = -\partial H / \partial x^i = \partial p_0 / \partial x^i$:

$$\begin{aligned} \frac{\partial}{\partial x^i}\left[g^{\mu\nu} p_\mu p_\nu = -m^2\right] \quad &\Longrightarrow \quad \frac{\partial g^{\mu\nu}}{\partial x^i} p_\mu p_\nu + 2 g^{\mu 0} p_\mu \frac{\partial p_0}{\partial x^i} = 0 \\ &\Longrightarrow \quad p^0\, \dot{p}_i = -\frac{1}{2}\frac{\partial g^{\mu\nu}}{\partial x^i} p_\mu p_\nu \ . \end{aligned} \tag{19}$$

To put this into a more familiar form, we express the derivative of $g^{\mu\nu}$ in terms of the derivative of its inverse, $g_{\mu\nu}$:

$$\frac{\partial g^{\mu\nu}}{\partial x^\lambda} = -g^{\mu\alpha} \frac{\partial g_{\alpha\beta}}{\partial x^\lambda} g^{\beta\nu} \ . \tag{20}$$

Eq. (19) then becomes

$$p^0 \dot{p}_i = \frac{1}{2} g^{\mu\alpha} \frac{\partial g_{\alpha\beta}}{\partial x^i} g^{\beta\nu} p_\mu p_\nu = \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^i} p^\mu p^\nu \ , \tag{21}$$

where in the last step we have changed the names of the summation indices. To see that this agrees with the expected result, simply replace each p^μ using Eq. (16):

$$\begin{aligned} m \frac{dt}{d\tau}\frac{d}{dt}\left(m\, g_{i\nu} \frac{dx^\nu}{d\tau}\right) &= \frac{1}{2} m^2 \frac{\partial g_{\mu\nu}}{\partial x^i} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \\ \Longrightarrow \quad \frac{d}{d\tau}\left(g_{i\lambda} \frac{dx^\lambda}{d\tau}\right) &= \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^i}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \ , \end{aligned} \tag{22}$$

which can easily be seen to agree with Eq. (3), where A is taken to be one.

PROBLEM 2: LORENTZ-INVARIANCE OF THE PHASE SPACE VOLUME IN SPECIAL RELATIVITY *(10 points)*†

Start out with the definition of the phase space density $\mathcal{N}$:

$$\text{Number of particles} = \mathcal{N} dx^1 dx^2 dx^3 dp_1 dp_2 dp_3 \tag{23}$$

Consider analyzing the system from two different inertial frames $\left(x^0, x^1, x^2, x^3\right)$ and $\left(x'^0, x'^1, x'^2, x'^3\right)$. Since the number of particles in a phase space volume is a physical quantity independent of the observer, we must have

$$\mathcal{N} d^3x d^3p = \mathcal{N}' d^3x' d^3p' \tag{24}$$

Now, we must find out how the the phase space volumes compare. Because of issues of simultaneity in both frames (different x^0 and x'^0) the solution through a direct Jacobian can be tricky. However, one can use the suggestion in the problem and first consider the case when one of the frames is the rest frame of the particles. Let the unprimed frame (x, p) be the rest frame and the primed frame (x', p') the boosted frame. Then a volume d^3x becomes Lorentz contracted in the boosted frame, yielding the relation

$$d^3x' = (1/\gamma) d^3x. \tag{25}$$

To relate the momentum volume element in the rest frame d^3p with that in the boosted frame d^3p', we can use the fact (used in Quantum Field Theory) that we can get a Lorentz invariant spatial momentum volume element by integrating the momentum four volume d^4p over a delta function enforcing the mass shell constraint:

$$\int d^4p\ \delta\left(p^\mu p_\mu + m^2\right) \ldots = \int \frac{d^3p}{2\sqrt{\boldsymbol{p}^2+m^2}} \ldots \tag{26}$$

The last equation is obtained by integrating the delta function over p_0. The Lorentz invariant momentum measure is then $\frac{d^3p}{2\sqrt{\boldsymbol{p}^2+m^2}}$ If we compare the momentum volume measures as seen in the rest frame, $\frac{d^3p}{2m}$ and in the boosted frame $\frac{d^3p'}{2\sqrt{\boldsymbol{p}'^2+m^2}}$ we find upon using the Lorentz invariance of the this momentum measure:

$$\frac{d^3p}{2m} = \frac{d^3p'}{2\sqrt{\boldsymbol{p}'^2+m^2}} \tag{27}$$

$$\Longrightarrow \quad d^3p' = \frac{\sqrt{\boldsymbol{p}'^2+m^2}}{m}\, d^3p$$

$$\Longrightarrow \quad d^3p' = \gamma\, d^3p. \tag{28}$$

In the last line, the relation $\gamma = \sqrt{\boldsymbol{p}'^2+m^2}/m$ was used. Combining (28) and (31) gives

$$d^3x' d^3p' = (1/\gamma) d^3x\, \gamma\, d^3p = d^3x\, d^3p \tag{29}$$

To prove the equality for the case of two phase volumes in generic Lorentz frames $d^3x'\, d^3p'$ and $d^3x''\, d^3p''$, you can use the fact that both frames are can be found from boosts -with the appropriate γ factors- from the rest frame we just used. So all that would be needed is to apply (32) for both generic frames, which straightforwardly yields $d^3x'\, d^3p' = d^3x''\, d^3p''$. Thus, the volumes cancel in (27) and we get:

$$\mathcal{N}(x^i, p_i, t) = \mathcal{N}'(x'^i, p'_i, t') \tag{30}$$

PROBLEM 3: GENERAL COORDINATE INVARIANCE OF THE PHASE SPACE VOLUME IN GENERAL RELATIVITY *(20 points)**

(a) We are asked to show that the density of particles in an arbitrary space with coordinates $\xi^1, \ldots, \xi^n$ can be written as

$$\rho(\xi^i, t) = \sum_\alpha \int \mathrm{d}\lambda\, \delta^n\big(\xi^i - \xi^i_\alpha(\lambda)\big)\, \delta\big(t - t_\alpha(\lambda)\big)\, \frac{\mathrm{d}t_\alpha}{\mathrm{d}\lambda}\ . \tag{31}$$

To show this, we start with the claim that $\rho(\xi^i, t)$ can be written as

$$\rho(\xi^i, t) = \sum_\alpha \delta^n\left(\xi^i - \xi^i_\alpha(t)\right)\ , \tag{32}$$

where $\xi^i_\alpha(t)$ is the ξ^i-coordinate of the α'th particle at time t. To verify this claim, note that the integral of this expression over an arbitrary volume is equal to the number of particles in that volume, which is exactly how a density is defined. To finish, recall that the delta function of a function of a variable can be evaluated by

$$\delta\big(f(\lambda)\big) = \sum_k \frac{\delta(\lambda - \lambda_k)}{\left|\dfrac{\mathrm{d}f}{\mathrm{d}\lambda}(\lambda_k)\right|}\ , \tag{33}$$

where the λ_k are the zeros of $f(\lambda)$, which are assumed to be simple zeros.‡ Applying this formula to $f(\lambda) = t - t_\alpha(\lambda)$, one has

$$\delta\big(t - t_\alpha(\lambda)\big) = \frac{\delta\big(\lambda - \lambda_\alpha(t)\big)}{\left|\dfrac{\mathrm{d}t_\alpha}{\mathrm{d}\lambda}\big(\lambda_\alpha(t)\big)\right|}\ , \tag{34}$$

where $\lambda_\alpha(t)$ is the value of λ for which $t_\alpha(\lambda) = t$. If this expression for $\delta\big(t - t_\alpha(\lambda)\big)$ is substituted into Eq. (31), one can integrate over λ to obtain Eq. (32), assuming that $t_\alpha(\lambda)$ is a monotonically increasing function of λ.

(b) The trajectories in the primed system are given by

$$X'^\mu(\lambda) = X'^\mu_c\big(X^\nu_\alpha(\lambda)\big)\ , \tag{35}$$

so the density function in primed coordinates is given by

$$\rho'(\xi'^i, t) = \sum_\alpha \int \mathrm{d}\lambda\, \delta^{n+1}\Big(X'^\mu - X'^\mu_c\big(X^\nu_\alpha(\lambda)\big)\Big)\, \frac{\mathrm{d}t'_\alpha}{\mathrm{d}\lambda}\ . \tag{36}$$

We now change variables inside the $(n+1)$-dimensional delta function, using the $(n+1)$-dimensional generalization of Eq. (33):

$$\delta^{n+1}\left(X'^\mu - X'^\mu_0\right) = \left|\mathrm{Det}\left(\frac{\partial X^\mu_c}{\partial X'^\nu}\right)\right|\ \delta^{n+1}\left(X^\mu - X^\mu_0\right)\ , \tag{37}$$

‡ This formula is well-known to those who know it, but it seems hard to find it in books. It is listed in the Wikipedia article on the "Dirac delta function," and it is given by J.D. Jackson in **Classical Electrodynamics**, 3rd Edition (Wiley, 1999), on p. 26. It can be verified by changing the variable of integration from λ to $z \equiv f(\lambda)$.

where $X_0'^\mu$ and X_0^μ are a set of constants related by $X_0'^\mu = X_c'^\mu(X_0^\nu)$. (Later we will let these "constants" depend on λ, and we will integrate over λ, but that does not prevent us from treating them as constants here.) Eq. (37) is also considered well-known by those who know it, but I have not been able to find it in print. (If you know a place where it appears in print, please let me know!) It is of course fine if you used it without explanation on your solutions, but for the sake of pedagogy I will show how it follows from the Jacobian transformation of integration volumes, combined with the fact that a delta function is really defined by the result of integrating with a test function.

Specifically, if $\varphi(X^\mu)$ is a test function, then $\delta^{n+1}(X^\mu - X_0^\mu)$ can be defined by

$$\int d^{n+1}X\, \delta^{n+1}(X^\mu - X_0^\mu)\, \varphi(X^\mu) \equiv \varphi(X_0^\mu)\ . \tag{38}$$

To change variables to X'^μ, we can start with the general relation

$$\int d^{n+1}X\, F(X^\mu) = \int d^{n+1}X' \left|\text{Det}\left(\frac{\partial X^\mu}{\partial X'^\nu}\right)\right| F(X^\mu)\big|_{X^\mu = X_c^\mu(X'^\nu)}\ , \tag{39}$$

where the vertical bar with the subscript indicates that F is to be evaluated at the coordinates X^μ that correspond under the coordinate transformation to X'^ν. Applying this general relation to the integral in Eq. (38),

$$\varphi(X_0^\mu) = \int d^{n+1}X\, \delta^{n+1}(X^\mu - X_0^\mu)\, \varphi(X^\mu) \tag{40a}$$

$$= \int d^{n+1}X' \left|\text{Det}\left(\frac{\partial X^\mu}{\partial X'^\nu}\right)\right| \left[\delta^{n+1}(X^\mu - X_0^\mu)\, \varphi(X^\mu)\right]\big|_{X^\mu = X_c^\mu(X'^\nu)} \tag{40b}$$

$$= \int d^{n+1}X'\, \delta^{n+1}\!\left(X'^\mu - X_0'^\mu\right)\, \varphi(X^\mu)\big|_{X^\mu = X_c^\mu(X'^\nu)}\ , \tag{40c}$$

where line (40c) is not obtained from the previous line, but rather by using the definition of the delta function, as in Eq. (38), to show that the integral is equal to $\varphi(X_0^\mu)$. By comparing line (40b) with line (40c), one sees that they agree if and only if Eq. (37) is valid. The "only-if" part of this statement requires that we know that line (40b) is equal to (40c) for all test functions φ, but that is the case. In a more formal mathematical setting, the test functions would be required from the beginning to belong to some specified class of well-behaved functions.

Using Eq. (37), Eq. (36) can be rewritten

$$\rho'(\xi'^i, t) = \sum_\alpha \int \mathrm{d}\lambda \left|\text{Det}\left(\frac{\partial X_c^\mu}{\partial X'^\nu}\right)\right| \delta^{n+1}\!\left(X^\mu - X_\alpha^\mu(\lambda)\right) \frac{\mathrm{d}t'_\alpha}{\mathrm{d}\lambda}\ . \tag{41}$$

To complete our task of expressing $\rho'(\xi'^i, t)$ in terms of $\rho(\xi^i, t)$, as given by Eq. (31), we need to rewrite $\frac{dt'_\alpha}{d\lambda}$. The rewriting is pretty simple, however, because as we follow a given particle, the variables λ, t, and t' are all redundant, with any one of them determining the other two. Thus, we can write

$$\frac{dt'_\alpha}{\mathrm{d}\lambda} = \frac{\mathrm{d}t'_\alpha}{\mathrm{d}t_\alpha}\frac{\mathrm{d}t_\alpha}{\mathrm{d}\lambda}\ . \tag{42}$$

Furthermore, since the velocity of a particle is uniquely determined by its position and time, $\frac{\mathrm{d}t'_\alpha}{\mathrm{d}t_\alpha}$ does not depend on which particle we are describing, but depends only on the coordinates ξ^i and t. Thus, the factor $\frac{\mathrm{d}t'_\alpha}{\mathrm{d}t_\alpha}$ can be written as $\frac{\mathrm{d}t'}{\mathrm{d}t}(\xi^i, t)$, and taken outside the sum and integral. Finally, Eq. (41) can be rewritten as

$$\rho'(\xi'^i, t) = \left|\text{Det}\left(\frac{\partial X_c^\mu}{\partial X'^\nu}\right)\right| \frac{\mathrm{d}t'}{\mathrm{d}t} \sum_\alpha \int \mathrm{d}\lambda\, \delta^{n+1}\!\left(X^\mu - X_\alpha^\mu(\lambda)\right) \frac{\mathrm{d}t_\alpha}{\mathrm{d}\lambda}\ , \tag{43}$$

which implies, making use of Eq. (31) and the convention that $X^i = \xi^i$ and $X^0 = t$, that

$$\boxed{\rho'(\xi'^i, t) = \left|\text{Det}\left(\frac{\partial X_c^\mu}{\partial X'^\nu}\right)\right| \frac{\mathrm{d}t'}{\mathrm{d}t}\, \rho(\xi^i, t)\ .} \tag{44}$$

The above formula is really the desired result, but when I made up the problem set I did not notice that it could be written this simply. To make contact with the formula given on the problem set, note that $\frac{\mathrm{d}t'}{\mathrm{d}t}$ can be rewritten by expanding it in the X^μ coordinates:

$$\frac{\mathrm{d}t'}{\mathrm{d}t} = \frac{\partial t'_c}{\partial X^\mu}\frac{\mathrm{d}X^\mu}{\mathrm{d}t} = \frac{\partial t'_c}{\partial t} + \frac{\partial t'_c}{\partial \xi^i}\frac{\mathrm{d}\xi^i}{\mathrm{d}t}\ . \tag{45}$$

(c) I will evaluate the determinant in Eq. (44) by using Eq. (21) of the problem set,

$$\text{Det}\left(\frac{\partial \xi_c^i}{\partial \xi'^j}\right) = \frac{\partial t'_c}{\partial t}\, \text{Det}\left(\frac{\partial X_c^\mu}{\partial X'^\nu}\right)\ . \tag{46}$$

However, the equations in the problem set express the primed coordinates in terms of the unprimed, so I will use the fact that

$$\text{Det}\left(\frac{\partial X_c^\mu}{\partial X'^\nu}\right) = \left[\text{Det}\left(\frac{\partial X_c'^\mu}{\partial X^\nu}\right)\right]^{-1}\ . \tag{47}$$

This equation can be shown by noting that the matrices on the two sides are inverses of each other. Eq. (46) can then be rewritten as

$$\text{Det}\left(\frac{\partial X_c^\mu}{\partial X'^\nu}\right) = \left[\text{Det}\left(\frac{\partial X_c'^\mu}{\partial X^\nu}\right)\right]^{-1} = \frac{\dfrac{\partial t_c}{\partial t'}}{\text{Det}\left(\dfrac{\partial \xi_c'^i}{\partial \xi^j}\right)} . \tag{48}$$

Note that $\text{Det}\left(\frac{\partial \xi_c'^i}{\partial \xi^j}\right)$ is the determinant of a 6×6 matrix that can be written in block form as

$$\text{Det}\left(\frac{\partial \xi_c'^i}{\partial \xi^j}\right) = \text{Det}\begin{pmatrix} \dfrac{\partial x'^i}{\partial x^j} & \dfrac{\partial x'^i}{\partial p_j} \\ \dfrac{\partial p_i'}{\partial x^j} & \dfrac{\partial p_i'}{\partial p_j} \end{pmatrix} . \tag{49}$$

The key equations are given as Eq. (28) of the problem set:

$$t' = x_c'^0(x^\nu) \equiv t_c'(x^\nu) , \tag{50a}$$

$$x'^i = x_c'^i(x^\nu) , \tag{50b}$$

$$p_i' = \frac{\partial x_c^\nu}{\partial x'^i} p_\nu \tag{50c}$$

$$= \frac{\partial x_c^j}{\partial x'^i} p_j + \frac{\partial t_c}{\partial x'^i} p_0(x^1, x^2, x^3, p_1, p_2, p_3, t) , \tag{50d}$$

so I will start here. From Eq. (50b) we see that

$$\frac{\partial x'^i}{\partial x^j} = \frac{\partial x_c'^i}{\partial x^j} , \quad \frac{\partial x'^i}{\partial p_j} = 0 . \tag{51}$$

Given the block form of Eq. (49), the vanishing of $\partial x'^i/\partial p_j$ means that the upper right 3×3 block vanishes, and therefore the lower left block is irrelevant, and

$$\text{Det}\left(\frac{\partial \xi_c'^i}{\partial \xi^j}\right) = \text{Det}\left(\frac{\partial x'^i}{\partial x^j}\right)\,\text{Det}\left(\frac{\partial p_i'}{\partial p^j}\right) . \tag{52}$$

The p' derivatives can be calculated from Eq. (50d), giving

$$\begin{aligned} \frac{\partial p_i'}{\partial p_j} &= \frac{\partial x_c^j}{\partial x'^i} + \frac{\partial t_c}{\partial x'^i}\frac{\partial p_0}{\partial p_j} \\ &= \frac{\partial x_c^j}{\partial x'^i} - \frac{\partial t}{\partial x'^i}\frac{\mathrm{d}x^j}{\mathrm{d}t} , \end{aligned} \tag{53}$$

where in the last step I used the fact that p_0 is the Hamiltonian, so its derivatives are related to the time derivatives of the canonical variables.

To evaluate Eq. (52), we can use the fact that the determinant of the product of two matrices is equal to the determinant of the product, so

$$\begin{aligned} \text{Det}\left(\frac{\partial \xi_c'^i}{\partial \xi^k}\right) &= \text{Det}\left(\frac{\partial x'^i}{\partial x^j}\frac{\partial p_k'}{\partial p_j}\right) \\ &= \text{Det}\left[\frac{\partial x_c'^i}{\partial x^j}\left(\frac{\partial x_c^j}{\partial x'^k} - \frac{\partial t_c}{\partial x'^k}\frac{\mathrm{d}x^j}{\mathrm{d}t}\right)\right] . \end{aligned} \tag{54}$$

To further simplify, we can make use of the fact that the spacetime coordinate transformation does not depend on the momenta (although it does affect the momenta). For that reason Eqs. (23–25) of the problem set can be written for the spacetime coordinates alone:

$$\frac{\partial \xi_c'^i}{\partial \xi^j}\frac{\partial x_c^j}{\partial x'^k} = \delta^i{}_k - \frac{\partial x_c'^i}{\partial t}\frac{\partial t_c}{\partial x'^k} . \tag{55}$$

$$\frac{\partial t_c'}{\partial x^j}\frac{\partial x_c^j}{\partial x'^k} = -\frac{\partial t_c'}{\partial t}\frac{\partial t_c}{\partial x'^k} . \tag{56}$$

$$\frac{\partial t_c'}{\partial x^j}\frac{\partial x_c^j}{\partial t'} = 1 - \frac{\partial t_c'}{\partial t}\frac{\partial t_c}{\partial t'} . \tag{57}$$

To evaluate the determinant in Eq. (54), we can expand the argument of the determinant in the 2nd line and then use identity (55) on the first term, finding

$$\text{Det}\left(\frac{\partial \xi_c'^i}{\partial \xi^k}\right) = \text{Det}\left[\delta_i^k - \frac{\partial t_c}{\partial x'^k}\left(\frac{\partial x_c'^i}{\partial t} + \frac{\partial x'^i}{\partial x^j}\frac{\mathrm{d}x^j}{\mathrm{d}t}\right)\right] . \tag{58}$$

The quantity in parentheses can be simplified as

$$\left(\frac{\partial x_c'^i}{\partial t} + \frac{\partial x'^i}{\partial x^j}\frac{\mathrm{d}x^j}{\mathrm{d}t}\right) = \frac{\mathrm{d}x'^i}{\mathrm{d}t} , \tag{59}$$

where the total derivative $\mathrm{d}x^i/\mathrm{d}t$ refers to the derivative along the trajectory of the particle, as in Eq. (45). Then

$$\text{Det}\left(\frac{\partial \xi_c'^i}{\partial \xi^k}\right) = \text{Det}\left[\delta_i^k - \frac{\partial t_c}{\partial x'^k}\frac{\mathrm{d}x'^i}{\mathrm{d}t}\right] , \tag{60}$$

which is ready for evaluation using the identity in Eq. (22) of the problem set,

$$\text{Det}\,(\delta_j^i + u^i v_j) = 1 + u^i v_i . \tag{61}$$

Thus,

$$\text{Det}\left(\frac{\partial \xi_c'^i}{\partial \xi^k}\right) = 1 - \frac{\partial t_c}{\partial x'^i}\frac{\mathrm{d}x'^i}{\mathrm{d}t} , \tag{60}$$

which can be simplified by using the chain rule relation

$$\frac{\partial t_c}{\partial x'^i}\frac{\mathrm{d}x'^i}{\mathrm{d}t} + \frac{\partial t_c}{\partial t'}\frac{\mathrm{d}t'}{\mathrm{d}t} = \frac{\partial t_c}{\partial X'^\mu}\frac{\mathrm{d}X'^\mu}{\mathrm{d}t} = \frac{\mathrm{d}t}{\mathrm{d}t} = 1 , \tag{61}$$

so finally

$$\text{Det}\left(\frac{\partial \xi_c'^i}{\partial \xi^k}\right) = \frac{\partial t_c}{\partial t'}\frac{\mathrm{d}t'}{\mathrm{d}t} . \tag{62}$$

By combining this result with Eqs. (44) and (48), one sees that

$$\boxed{\rho'(\xi'^i, t) = \rho(\xi^i, t) ,} \tag{63}$$

which was the ultimate goal of this problem.

PROBLEM 4: SPECIFIC INTENSITY *(10 points)**

By the definition of specific intensity I_ν, the energy dE hitting a detector of area dA during a time dt, from a solid angle dΩ and within a frequency interval dν, is given by

$$\mathrm{d}E = I_\nu \,\mathrm{d}A\,\mathrm{d}t\,\mathrm{d}\Omega\,\mathrm{d}\nu \quad . \tag{64}$$

Since photons have energy $h\nu$, the number of photons is given by

$$\mathrm{d}N = \frac{\mathrm{d}E}{h\nu} = \frac{I_\nu}{h\nu}\,\mathrm{d}A\,\mathrm{d}t\,\mathrm{d}\Omega\,\mathrm{d}\nu . \tag{65}$$

During the time interval dt the photons travel a distance $\mathrm{d}\ell = c\,\mathrm{d}t$, and since they hit a detector of area dA, the volume containing the photons is

$$\mathrm{d}^3x = \mathrm{d}\ell\,\mathrm{d}A = c\,\mathrm{d}t\,\mathrm{d}A . \tag{66}$$

The magnitude of the photon momentum is

$$p = \frac{E}{c} = \frac{h\nu}{c} , \tag{67}$$

so the momentum space volume is

$$\mathrm{d}^3p = p^2\,\mathrm{d}\Omega\,\mathrm{d}p = \left(\frac{h\nu}{c}\right)^2 \mathrm{d}\Omega \left(\frac{h}{c}\right)\mathrm{d}\nu . \tag{68}$$

Putting these results together,

$$\mathrm{d}N = \frac{I_\nu}{h\nu}\frac{1}{c}\left(\frac{c}{h\nu}\right)^2\left(\frac{c}{h}\right)\mathrm{d}^3x\,\mathrm{d}^3p = \frac{c^2}{h^4\nu^3}I_\nu \mathrm{d}^3x\,\mathrm{d}^3p = \frac{c^2}{(2\pi\hbar)^4}\frac{I_\nu}{\nu^3}\,\mathrm{d}^3x\,\mathrm{d}^3p . \tag{69}$$

Thus, the phase space density is

$$\boxed{\mathcal{N}_\gamma = \frac{c^2}{(2\pi\hbar)^4}\frac{I_\nu}{\nu^3} ,} \tag{70}$$

as claimed.

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†Solution written by Carlos Santana.

*Solution written by Alan Guth.