MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.952: Particle Physics of the Early Universe — May 23, 2009
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PROBLEM SET 5 SOLUTIONS

PROBLEM 1: EINSTEIN EQUATIONS IN SYNCHRONOUS GAUGE

(15 points)¶

For this problem, we begin by using the expression for the perturbation of the Ricci tensor's spatial components δR_{jk} in Weinberg's equation (5.1.13):

$$\begin{aligned}\delta R_{jk} = & -\frac{1}{2}\partial_j\partial_k h_{00} - \left(2\dot{a}^2 + a\ddot{a}\right)\delta_{jk}h_{00} - \frac{1}{2}a\dot{a}\delta_{jk}\dot{h}_{00} \\ & + \frac{1}{2a^2}\left(\nabla^2 h_{jk} - \partial_i\partial_j h_{ik} - \partial_i\partial_k h_{ij} + \partial_j\partial_k h_{ii}\right) \\ & - \frac{1}{2}\ddot{h}_{jk} + \frac{\dot{a}}{2a}\left(\dot{h}_{jk} - \delta_{jk}\dot{h}_{ii}\right) + \frac{\dot{a}^2}{a^2}\left(-2h_{jk} + \delta_{jk}h_{ii}\right) + \frac{\dot{a}}{a}\delta_{jk}\partial_i h_{i0} \\ & + \frac{1}{2}\left(\partial_j\dot{h}_{k0} + \partial_k\dot{h}_{j0}\right) + \frac{\dot{a}}{2a}\left(\partial_j h_{k0} + \partial_k h_{j0}\right).\end{aligned} \tag{1}$$

In the synchronous gauge, E and F are set to zero in the decomposition of the metric perturbation h_{jk} into scalar, vector and tensor modes. As indicated by the problem, we will not include the vector modes so we set $C_j = 0$ and $G_j = 0$. The metric perturbation $h_{\mu\nu}$ then takes the form:

$$\begin{aligned} h_{00} &= h_{j0} = 0 \\ h_{jk} &= a^2\left[A\delta_{jk} + \partial_j\partial_k B + D_{jk}\right], \end{aligned} \tag{2}$$

where D_{jk} is a symmetric tensor that satisfies $D_{ii} = 0$ and $\partial_j D_{jk} = 0$. Since $h_{00} = h_{j0} = 0$, the perturbation of the Ricci tensor simplifies somewhat to

$$\begin{aligned}\delta R_{jk} = & \frac{1}{2a^2}\left(\nabla^2 h_{jk} - \partial_i\partial_j h_{ik} - \partial_i\partial_k h_{ij} + \partial_j\partial_k h_{ii}\right) \\ & - \frac{1}{2}\ddot{h}_{jk} + \frac{\dot{a}}{2a}\left(\dot{h}_{jk} - \delta_{jk}\dot{h}_{ii}\right) + \frac{\dot{a}^2}{a^2}\left(-2h_{jk} + \delta_{jk}h_{ii}\right)\end{aligned} \tag{3}$$

Inserting (2) into the first line of equation (3) gives after some algebra:

$$\begin{aligned} & \frac{1}{2a^2}\left(\nabla^2 h_{jk} - \partial_i\partial_j h_{ik} - \partial_i\partial_k h_{ij} + \partial_j\partial_k h_{ii}\right) = \\ & \qquad \frac{1}{2}\left(\delta_{jk}\nabla^2 A + \nabla^2 D_{jk} + \partial_j\partial_k A\right).\end{aligned} \tag{4}$$

Similarly, the second line of equation (3) gives:

$$\begin{aligned} & -\frac{1}{2}\ddot{h}_{jk} + \frac{\dot{a}}{2a}\left(\dot{h}_{jk} - \delta_{jk}\dot{h}_{ii}\right) + \frac{\dot{a}^2}{a^2}\left(-2h_{jk} + \delta_{jk}h_{ii}\right) = \\ & \qquad - \delta_{jk}\left[\left(a\ddot{a} + 2\dot{a}^2\right)A + 3a\dot{a}\dot{A} + \frac{1}{2}a^2\ddot{A} + \frac{a\dot{a}}{2}\nabla^2\dot{B}\right] \\ & \qquad - \partial_j\partial_k\left[\left(a\ddot{a} + 2\dot{a}^2\right)B + \frac{3}{2}a\dot{a}\dot{B} + \frac{1}{2}a^2\ddot{B}\right] \\ & \qquad - \left(a\ddot{a} + 2\dot{a}^2\right)D_{jk} - \frac{3}{2}a\dot{a}\dot{D}_{jk} - \frac{1}{2}a^2\ddot{D}_{jk}\end{aligned} \tag{5}$$

Thus δR_{jk} is expressed in terms of the scalar and tensor perturbations as

$$\begin{aligned}\delta R_{jk} = & \delta_{jk}\left[\frac{1}{2}\nabla^2 A - \left(a\ddot{a} + 2\dot{a}^2\right)A - 3a\dot{a}\dot{A} - \frac{1}{2}a^2\ddot{A} - \frac{a\dot{a}}{2}\nabla^2\dot{B}\right] \\ & + \partial_j\partial_k\left[\frac{1}{2}A - \left(a\ddot{a} + 2\dot{a}^2\right)B - \frac{3}{2}a\dot{a}\dot{B} - \frac{1}{2}a^2\ddot{B}\right] \\ & + \frac{1}{2}\nabla^2 D_{jk} - \left(a\ddot{a} + 2\dot{a}^2\right)D_{jk} - \frac{3}{2}a\dot{a}\dot{D}_{jk} - \frac{1}{2}a^2\ddot{D}_{jk}\end{aligned} \tag{6}$$

Now we use this expression in Einstein's equations $R_{\mu\nu} = -8\pi G S_{\mu\nu} = -8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\lambda{}_\lambda\right)$. We decompose $T_{\mu\nu}$ in terms of the unperturbed, perfect fluid part $\bar{T}_{\mu\nu}$ and the correction $\delta T_{\mu\nu}$. The latter has the spatial components — neglecting vector modes — given by

$$\delta T_{jk} = \bar{p}\,h_{jk} + a^2\left[\delta_{jk}\delta p + \partial_j\partial_k\pi^S + \pi^T_{jk}\right] \tag{7}$$

with $\bar{p}$ the pressure in the unperturbed FRW universe, δp the pressure perturbation and with π^T_{jk} satisfying $\pi^T_{ii} = 0$ and $\partial_j\pi^T_{jk} = 0$.

To first order in the perturbations, the purely spatial components of the Einstein equation yield

$$\delta R_{jk} = -8\pi G\left(\delta T_{jk} - \frac{1}{2}h_{jk}\bar{T}^\lambda{}_\lambda - \frac{1}{2}\bar{g}_{jk}\delta T^\lambda{}_\lambda\right). \tag{8}$$

We shall assume that the unperturbed metric is the $K = 0$ Robertson-Walker universe. From equation (5.1.43) in Weinberg's text, we find $\delta T^\lambda{}_\lambda = 3\delta p - \delta\rho + \nabla^2\pi^S$. Similarly, we can find the trace $\bar{T}^\lambda{}_\lambda$ in terms of the scale factor a as

$$\bar{T}^\lambda{}_\lambda = -\frac{3}{4\pi G}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right), \tag{9}$$

and also

$$\bar{p}(t) = -\frac{1}{8\pi G}\left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) \ . \tag{10}$$

Using this information, the jk component of Einstein's equations becomes

$$\begin{aligned}\delta R_{jk} = -8\pi G a^2 &\left[\partial_j\partial_k\pi^S + \pi^T_{jk} + \frac{1}{2}\delta_{jk}(\delta\rho - \delta p - \nabla^2\pi^S)\right] \\ &- \left(\frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2}\right) h_{jk} \ .\end{aligned} \tag{11}$$

Or, upon using $h_{jk} = a^2\left[A\delta_{jk} + \partial_j\partial_k B + D_{jk}\right]$, equation (11) becomes

$$\begin{aligned}\delta R_{jk} = -8\pi G &\left[\partial_j\partial_k\pi^S + \pi^T_{jk} + \frac{1}{2}\delta_{jk}\left(\delta\rho - \delta p - \nabla^2\pi^S\right)\right] \\ &- (a\ddot{a} + 2\dot{a}^2)(A\,\delta_{jk} + \partial_j\partial_k B + D_{jk}) \ .\end{aligned} \tag{12}$$

Upon setting the right-hand sides of equations (6) and (12) equal to each other, we obtain the final form of the δR_{jk} equation:

$$\boxed{\begin{aligned}&\frac{1}{2}\partial_j\partial_k A - \frac{1}{2}a^2\partial_j\partial_k\ddot{B} - \frac{3}{2}a\dot{a}\partial_j\partial_k\dot{B} - \frac{1}{2}a^2\partial_j\partial_k\ddot{D} - \frac{3}{2}a\dot{a}\dot{D}_{jk} + \frac{1}{2}\nabla^2 D_{jk} \\ &\quad + \delta_{jk}\left(\frac{1}{2}a^2\ddot{A} + 3a\dot{a}\dot{A} - \frac{1}{2}\nabla^2 A + \frac{1}{2}a\dot{a}\nabla^2\dot{B}\right) \\ &\quad\quad = -8\pi G a^2\left[\partial_j\partial_k\pi^S + \pi^T_{jk} + \frac{1}{2}\delta_{jk}\left(\delta p - \delta\rho + \nabla^2\pi^S\right)\right] \ .\end{aligned}} \tag{13}$$

This equation has the generic form

$$\partial_j\partial_k X + \delta_{jk} Y + Z_{jk} = 0 \ , \tag{14}$$

where $Z_{ii} \equiv 0$ and $\partial_j Z_{jk} \equiv 0$. As long as X, Y, and Z are Fourier expandable, so that the equation can be rewritten in Fourier space as

$$-q_i q_j X_q + \delta_{jk} Y_q + Z_{jk,q} = 0 \ , \tag{15}$$

with $Z_{ii,q} \equiv 0$ and $q_j Z_{jk,q} \equiv 0$, then one can show that X, Y, and Z must vanish separately. This is the case for cosmologically interesting density perturbations. However, it is worth pointing out that this decomposition depends on boundary conditions, and is not completely general. If for example we allow X to equal $\frac{1}{2}\omega_{ij}x^i x^j$, then $\partial_j\partial_k X = \omega_{ij}$. If ω_{ij} contains a piece proportional to δ_{ij}, then the X term will mix with the Y term in Eq. (14). If ω_{ij} contains a traceless piece, it will mix with the Z term. Terms like these would arise, for example, if one described a slightly open or slightly closed Robertson-Walker universe as a perturbation of a flat universe. The perturbative description would break down at large distances from the origin, but it can still be a perfectly valid description in some finite region about the origin.

In any case, we are interested here in cosmological perturbations which are described by a Fourier expansion, so the three contributions in Eq. (13) must vanish separately. Looking first at the piece proportional to δ_{jk}, we find

$$\boxed{-4\pi G a^2\left(\delta\rho - \delta p - \nabla^2\pi^S\right) = \frac{1}{2}\nabla^2 A - \frac{1}{2}a^2\ddot{A} - 3a\dot{a}\dot{A} - \frac{a\dot{a}}{2}\nabla^2\dot{B} \ ,} \tag{16}$$

which is Weinberg's equation (5.3.28). By insisting that the terms involving $\partial_j\partial_k$ should vanish, one finds

$$\boxed{\partial_j\partial_k\left[-16\pi G a^2\pi^S\right] = \partial_j\partial_k\left[A - a^2\ddot{B} - 3a\dot{a}\dot{B}\right] \ ,} \tag{17}$$

which is Weinberg's equation (5.3.29) for the non-zero modes. Finally, although you were not asked to write this equation, we can extract the traceless and divergenceless piece of Eq. (13),

$$\boxed{-16\pi G a^2\pi^T_{jk} = \nabla^2 D_{jk} - a^2\ddot{D}_{jk} - 3a\dot{a}\dot{D}_{jk} \ ,} \tag{18}$$

which corresponds to Weinberg's equation (5.1.53).

PROBLEM 2: HOMOGENEOUS GAUGE TRANSFORMATIONS IN SYNCHRONOUS GAUGE *(10 points)*†

For the synchronous gauge, the general first-order spatially homogeneous scalar and tensor perturbations to the metric take the form:

$$h_{00} = 0 \tag{19}$$

$$h_{i0} = 0 \tag{20}$$

$$h_{ij} = a^2\left[A(t)\delta_{ij} + D_{ij}(t)\right] \tag{21}$$

Now consider the gauge transformations in equations (5.3.5) through (5.3.7) in Weinberg's text.

$$\Delta h_{00} = -2\frac{\partial \epsilon_0}{\partial t} \tag{22}$$

$$\Delta h_{i0} = -\frac{\partial \epsilon_i}{\partial t} - \frac{\partial \epsilon_0}{\partial x^i} + 2\frac{\dot{a}}{a}\epsilon_i \tag{23}$$

$$\Delta h_{ij} = -\frac{\partial \epsilon_i}{\partial x^j} - \frac{\partial \epsilon_j}{\partial x^i} + 2a\dot{a}\delta_{ij}\epsilon_0. \tag{24}$$

To preserve the synchronous gauge condition for h_{00}, we must have $\Delta h_{00} = 0$ in equation (22), which implies

$$\begin{aligned} \Delta h_{00} &= -2\frac{\partial \epsilon_0}{\partial t} = 0 \\ \implies \quad \epsilon_0 &= \epsilon(\boldsymbol{x}). \end{aligned} \tag{25}$$

Continuing with equation (23), we must also have $\Delta h_{i0} = 0$ to preserve the synchronous gauge condition, so that

$$\begin{aligned} \Delta h_{i0} &= -\frac{\partial \epsilon_i}{\partial t} - \frac{\partial \epsilon(\boldsymbol{x})}{\partial x^i} + 2\frac{\dot{a}}{a}\epsilon_i = 0 \\ \implies \quad & \frac{\partial \epsilon_i(\boldsymbol{x},t)}{\partial t} - 2\frac{\dot{a}}{a}\epsilon_i(\boldsymbol{x},t) = -\frac{\partial \epsilon(\boldsymbol{x})}{\partial x^i} \end{aligned} \tag{26}$$

This last equation can be easily solved using an integrating factor $\mu(t) = \exp\left(\int \left(-2\frac{\dot{a}}{a}\right) dt\right) = a(t)^{-2}$ giving us

$$\epsilon_i(\boldsymbol{x},t) = a(t)^2\alpha_i(\boldsymbol{x}) - a(t)^2\frac{\partial \epsilon(\boldsymbol{x})}{\partial x^i}\int_{\mathcal{T}}^t \frac{dt'}{a(t')^2}, \tag{27}$$

where $\alpha_i(\boldsymbol{x})$ is an arbitrary vector function, to be determined by enforcing homogeneity.

The last gauge transformation, that of h_{ij}, can be written using our results for ϵ_0 and ϵ_i as

$$\begin{aligned} \Delta h_{ij} &= -\frac{\partial \epsilon_i}{\partial x^j} - \frac{\partial \epsilon_j}{\partial x^i} + 2a\dot{a}\delta_{ij}\epsilon_0 \\ &= -\left[a^2\frac{\partial \alpha_i(\boldsymbol{x})}{\partial x^j} - a^2\frac{\partial^2 \epsilon(\boldsymbol{x})}{\partial x^j \partial x^i}\int_{\mathcal{T}}^t \frac{dt'}{a(t')^2}\right] \\ &\quad - \left[a^2\frac{\partial \alpha_j(\boldsymbol{x})}{\partial x^i} - a^2\frac{\partial^2 \epsilon(\boldsymbol{x})}{\partial x^i \partial x^j}\int_{\mathcal{T}}^t \frac{dt'}{a(t')^2}\right] + 2a\dot{a}\delta_{ij}\epsilon(\boldsymbol{x}) \\ &= -a^2\left(\frac{\partial \alpha_i(\boldsymbol{x})}{\partial x^j} + \frac{\partial \alpha_j(\boldsymbol{x})}{\partial x^i}\right) + 2a^2\frac{\partial^2 \epsilon(\boldsymbol{x})}{\partial x^i \partial x^j}\int_{\mathcal{T}}^t \frac{dt'}{a(t')^2} + 2a\dot{a}\delta_{ij}\epsilon(\boldsymbol{x}). \end{aligned} \tag{28}$$

Now, by enforcing $\boldsymbol{x}$-independence of Δh_{ij}, we must take $\epsilon(\boldsymbol{x}) = \epsilon =$ constant. Also, we take $\alpha_i(\boldsymbol{x}) = \omega_{ij}x^j$, with ω_{ij} a constant matrix. With these choices, we find

$$\Delta h_{ij} = -a^2\left(\omega_{ij} + \omega_{ji}\right) + 2a\dot{a}\delta_{ij}\epsilon \tag{29}$$

$$= \delta_{ij}\left(-\frac{2}{3}a^2\omega_{kk} + 2a\dot{a}\epsilon\right) - a^2\left(\omega_{ij} + \omega_{ji} - \frac{2}{3}\omega_{kk}\delta_{ij}\right), \tag{30}$$

where in the last line we separated the symmetric, traceless part from that proportional to δ_{ij}. However, using the equations for h_{ij} at the beginning of this problem — which give h_{ij} in terms of the scalar perturbation A and the tensor D_{ij} — we can see that

$$\Delta h_{ij} = a^2\delta_{ij}\Delta A + a^2\Delta D_{ij}. \tag{31}$$

Upon comparing the parts proprtional to δ_{ij} and the symmetric, traceless tensor parts in (30) and (31) we find:

$$\Delta A = -\frac{2}{3}\omega_{kk} + 2H\epsilon, \tag{32}$$

$$\Delta D_{ij} = -\left(\omega_{ij} + \omega_{ji} - \frac{2}{3}\omega_{kk}\,\delta_{ij}\right). \tag{33}$$

We can also find the corresponding expressions for the changes in δp, $\delta\rho$, δu and π^S using the expressions for the gauge transformations in equations (5.3.14) and (5.3.15) in Weinberg's book, together with our expressions for the gauge functions ϵ_0 and ϵ_i:

$$\Delta\delta p = \dot{\bar{p}}\epsilon_0 = \dot{\bar{p}}\epsilon, \tag{34}$$

$$\Delta\delta\rho = \dot{\bar{\rho}}\epsilon_0 = \dot{\bar{\rho}}\epsilon, \tag{35}$$

$$\Delta\delta u = -\epsilon_0 = -\epsilon, \tag{36}$$

$$\Delta\pi^S = 0. \tag{37}$$

Now, since $\{h_{\mu\nu}, T_{\mu\nu}\}$ and $\{h_{\mu\nu} + \Delta h_{\mu\nu}, T_{\mu\nu} + \Delta T_{\mu\nu}\}$ are both solutions to the field equations and conservation equations, their difference must also be a solution. Thus there is always a spatially homogeneous solution of the synchronous gauge field and conservation equations with:

$$\boxed{\begin{aligned} A &= -\frac{2}{3}\omega_{kk} + 2H\epsilon, &(38)\\ D_{ij} &= -\left(\omega_{ij} + \omega_{ji} - \frac{2}{3}\omega_{kk}\,\delta_{ij}\right) &(39)\\ \delta p &= \dot{\bar{p}}\epsilon, &(40)\\ \delta\rho &= \dot{\bar{\rho}}\epsilon, &(41)\\ \delta u &= -\epsilon, &(42)\\ \pi^S &= 0. &(43)\end{aligned}}$$

PROBLEM 3: CONSTRUCTION OF ADIABATIC SOLUTIONS *(10 points)**

Weinberg's Eq. (5.3.30), when written in full, becomes $8\pi G a(\bar\rho + \bar p)\partial_i\,\delta u = a\partial_i \dot{A}$, where the dot over the A was incorrectly omitted in the problem set. For the solution described in Problem 2, $A = -\frac{2}{3}\omega_{kk} + 2H\epsilon$, and $\delta u = -\epsilon$, as described in Eqs. (38) and (42). Using $\dot{H} = -4\pi G(\bar\rho + \bar p)$, it can be seen that this equation is satisfied.

The other constraint equation is (5.3.29), the detailed form of which was derived in Problem 1 as Eq. (17). For the adiabatic solutions $\pi^S = 0$, so for nonzero q the Fourier space representation of Eq. (17) becomes

$$\begin{aligned} &A - a^2\ddot{B} - 3a\dot{a}\dot{B} = 0\\ &\quad\Longrightarrow\quad \frac{1}{a}\frac{\mathrm{d}}{\mathrm{d}t}\left(a^3\dot{B}\right) = A = 2\frac{\dot a}{a}\epsilon - \frac{2}{3}\omega_{kk}\\ &\quad\Longrightarrow\quad \frac{\mathrm{d}}{\mathrm{d}t}\left(a^3\dot{B}\right) = 2\epsilon\dot{a} - \frac{2}{3}\omega_{kk}a\\ &\quad\Longrightarrow\quad a^3\dot{B} = 2\epsilon a - \frac{2}{3}\omega_{kk}\int_{\mathcal{T}}^t a(t')\,\mathrm{d}t' + \mathcal{B}\ , \end{aligned} \tag{44}$$

where $\mathcal{B}$ is a constant of integration. Continuing,

$$\begin{aligned} &\dot{B} = \frac{2\epsilon}{a^2} - \frac{2}{3}\frac{\omega_{kk}}{a^3}\int_{\mathcal{T}}^t a(t')\,\mathrm{d}t' + \frac{\mathcal{B}}{a^3}\\ &\quad\Longrightarrow\quad B = 2\epsilon\int_{\mathcal{T}}^t \frac{\mathrm{d}t'}{a^2(t')} - \frac{2}{3}\omega_{kk}\int_{\mathcal{T}}^t \frac{\mathrm{d}t'}{a^3(t')}\int_{\mathcal{T}}^{t'} a(t'')\,\mathrm{d}t'' + \mathcal{B}\int_{\mathcal{T}}^t \frac{\mathrm{d}t'}{a^3(t')}\ . \end{aligned} \tag{45}$$

At this point one might notice a subtle point: the mode proportional to ϵ is exactly the residual gauge freedom of synchronous gauge, which Weinberg describes in Eqs. (5.3.40)–(5.3.42). Thus this mode has no physical significance, even when the $e^{i\boldsymbol{q}\cdot\boldsymbol{x}}$ spatial dependence is included, and so we will drop it. The two physical modes are the one proportional to ω_{kk} and the one proportional to $\mathcal{B}$. Note that the $\mathcal{B}$ mode has $A = 0$, and that both modes have no perturbations in the energy-momentum tensor variables. Thus, the new description of the adiabatic modes is given by

$$\boxed{\begin{aligned} A &= -\frac{2}{3}\omega_{kk}\ , &(46)\\ B &= -\frac{2}{3}\omega_{kk}\int_{\mathcal{T}}^t \frac{\mathrm{d}t'}{a^3(t')}\int_{\mathcal{T}}^{t'} a(t'')\,\mathrm{d}t'' + \mathcal{B}\int_{\mathcal{T}}^t \frac{\mathrm{d}t'}{a^3(t')}\ , &(47)\\ \delta p &= \delta\rho = \delta u = \pi^S = 0\ . &(48)\end{aligned}}$$

The statement in the Problem Set that B can be found by using Eq. (5.3.13) seems to have been slightly exaggerated. Eq. (5.3.13) says that

$$\Delta F = \frac{1}{a}\left(-\epsilon_0 - \dot{\epsilon}^S + \frac{2\dot a}{a}\epsilon^S\right)\ , \tag{49}$$

where in this case $\epsilon_0 = \epsilon =$ const. This equation can be solved by noting that $\Delta F = 0$ can be written as

$$a^2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\epsilon^S}{a^2}\right) = -\epsilon \quad\Longrightarrow\quad \frac{\epsilon^S}{a^2} = -\epsilon\int_{\mathcal{T}}^t \frac{\mathrm{d}t'}{a^2(t')}\ . \tag{50}$$

Eq. (5.3.13) also tells us that $B = -2\epsilon^S/a^2$, so this calculation reproduces the first term in Eq. (45), the term that we dropped because it is purely a gauge mode. It does not appear to be possible to generate either of the physical modes in this way. One should probably not be surprised that this method fails to generate the full answer, since the boundary conditions used in these homogeneous solutions, with $\epsilon_i \propto \omega_{ij}x^j$, are loose enough so that the decomposition used in Eq. (5.3.13) is not unique.

Note that in integrating $\dot{B}$ to obtain B in Eq. (45), one could have added a constant of integration to the answer, corresponding to a time-independent contribution to B. However, if one looks at the synchronous gauge equations of motion in Eqs. (5.3.28)–(5.3.33), one sees that B **always** appears as $\dot{B}$ or $\ddot{B}$. Thus a time-independent contribution to B suspiciously has no effect on any of the other variables. It turns out that changing B by a time-independent function is another residual gauge freedom of synchronous gauge, one that is not discussed in Weinberg's book. If one thinks of synchronous gauge as an evolution of equal-time slices, the residual gauge freedom that Weinberg discusses corresponds to choosing an initial slice that is slightly offset in the time direction from the original choice. The other residual gauge freedom is to perturb the spatial coordinate system on the initial slice. In terms of the formalism of Eq. (5.3.13), it corresponds to a gauge transformation with e^S/a^2 chosen to be independent of time, which can be seen to produce a time-independent change in B with no change in the other variables.

In the next problem we will see how B can also be determined by gauge transforming from the Newtonian gauge.

PROBLEM 4: GAUGE EQUIVALENCE OF THE ADIABATIC SOLUTIONS IN SYNCHRONOUS AND NEWTONIAN GAUGES *(10 points)**

Let us transform from Newtonian gauge to synchronous gauge, starting with the solution described by Weinberg's Eqs. (5.4.14–5.4.18). To avoid confusion with the gauge transformation that we need to describe, I will add a subscript to $\epsilon(t)$ in Eq. (5.4.15), writing it as

$$\epsilon_{\mathcal{R}} = \frac{\mathcal{R}}{a(t)} \int_{\mathcal{T}}^{t} a(t')\,\mathrm{d}t' \ , \tag{51}$$

where $\mathcal{R} = \frac{1}{3}\omega_{kk}$ and

$$\Psi = \Phi = -\dot{\epsilon}_{\mathcal{R}} \ . \tag{52}$$

Note that these equations describe the adiabatic solution, which due to the implicit $e^{i\boldsymbol{q}\cdot\boldsymbol{x}}$ spatial dependence is not a gauge transformation of the homogenous solution. We now wish to find a genuine gauge transformation that takes this solution to synchronous gauge.

Following Weinberg's description starting with Eq. (5.3.44), we seek a function ϵ_0 satisfying

$$\dot{\epsilon}_0 = -\Phi \ . \tag{53}$$

Clearly,

$$\epsilon_0 = \epsilon_{\mathcal{R}} \tag{54}$$

satisfies the desired equation. Then, according to Eq. (5.3.46), A is given by

$$A = -2\Psi + 2H\epsilon_0 \ , \tag{55}$$

which gives

$$\begin{aligned} A &= 2\dot{\epsilon}_{\mathcal{R}} + 2H\epsilon_{\mathcal{R}} \\ &= 2\mathcal{R}\left[\left(1 - \frac{H}{a(t)}\int_{\mathcal{T}}^{t} a(t')\,\mathrm{d}t'\right) + \frac{H}{a(t)}\int_{\mathcal{T}}^{t} a(t')\,\mathrm{d}t'\right] \\ &= 2\mathcal{R} = \frac{2}{3}\omega_{kk} \ . \end{aligned} \tag{56}$$

This agrees with the expression for A in Eq. (46), assuming that the synchronous gauge solution constructed in Problem 3 is based on the opposite sign choice for the meaning of ω_{ij}. To find the value of B that results from the gauge transformation, we must solve

$$\frac{\partial}{\partial t}\left(\frac{\epsilon^S}{a^2}\right) = -\frac{\epsilon_0}{a^2} = -\frac{\epsilon_{\mathcal{R}}}{a^2} \ . \tag{57}$$

Using $B = -\frac{2}{a^2}\epsilon^S$ and integrating, we find

$$B = 2\int_{\mathcal{T}}^{t} \frac{\epsilon_{\mathcal{R}}\mathrm{d}t'}{a^2(t')} = \frac{2}{3}\omega_{kk}\int_{\mathcal{T}}^{t} \frac{\mathrm{d}t'}{a^3(t')}\int_{\mathcal{T}}^{t'} a(t'')\,\mathrm{d}t''. \tag{58}$$

This matches the 1st term in Eq. (47), again allowing for a sign change in the meaning of ω_{ij}. By looking at the gauge transformation equations (5.3.14),

$$\Delta\delta p = \dot{\bar{p}}\,\epsilon_0 \ , \quad \Delta\delta\rho = \dot{\bar{\rho}}\,\epsilon_0 \ , \quad \Delta\delta u = -\epsilon_0 \ , \tag{59}$$

one sees that values of δp, $\delta\rho$, and δu in Eqs. (40)–(42) are canceled by the gauge transformation, so that they vanish in synchronous gauge.

To find the other $\mathcal{B}$ term in the synchronous gauge solution, we need to start with the 2nd adiabatic solution in Newtonian gauge, the one described by Eqs. (5.4.19–5.4.21). Here the solution can be written

$$\Psi = \Phi = -\dot{\epsilon}_C \ , \tag{60}$$

where

$$\epsilon_{\mathcal{C}} = \frac{C}{a(t)} \ , \ \text{so} \quad \Psi = \Phi = \frac{\mathcal{C}H}{a(t)} \ . \tag{61}$$

So the gauge transformation uses $\epsilon_0 = \epsilon_{\mathcal{C}}$, and

$$A = -2\Psi + 2H\epsilon_C = 0 \ . \tag{62}$$

To find B for this solution, we solve

$$\frac{\partial}{\partial t}\left(\frac{\epsilon^S}{a^2}\right) = -\frac{\epsilon_C}{a^2} = -\frac{\mathcal{C}}{a^3} \ , \tag{63}$$

and then

$$B = -\frac{2}{a^2}\epsilon^S = 2\mathcal{C}\int_{\mathcal{T}}^{t} \frac{\mathrm{d}t'}{a^3(t')} \ . \tag{64}$$

This agrees with Eq. (47), if we take the arbitrary constant $\mathcal{C}$ to equal $\mathcal{B}/2$. The $\mathcal{C}$ solution has

$$\frac{\delta\rho}{\dot{\rho}} = \frac{\delta p}{\dot{p}} = -\delta u = -\frac{\mathcal{C}}{a(t)} \ , \tag{65}$$

and it can be seen that the gauge transformation will lead to these quantities vanishing in synchronous gauge.

¶Solution written by Carlos Santana and Alan Guth.

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