Basic Notation

∪  \( A \cup B \) denotes the union of sets \( A \) and \( B \)
∩  \( A \cap B \) denotes the intersection of the sets \( A \) and \( B \)
⊂  \( A \subset B \) denotes that \( A \) is a subset of \( B \).
    (May or may not mean proper subset.)
−  \( B - A \) denotes the complement in \( B \) of the set \( A \)
∈  \( p \in A \) denotes that \( p \) is an element of \( A \)
\{\}  \( \{p \in A | Q\} \) denotes the set consisting of those elements
    \( p \) of the set \( A \) which satisfy condition \( Q \)
×  Cartesian product; \( A \times B \) is the set \( \{(a, b) | a \in A \text{ and } b \in B\} \)
∅  the empty set

\( \mathbb{R} \) the set of real numbers
\( \mathbb{R}^n \) the set of \( n \)-tuples of real numbers
\( \mathbb{C} \) the set of complex numbers
\( \mathbb{C}^n \) the set of \( n \)-tuples of complex numbers
\( f : A \rightarrow B \) denotes that \( f \) is a map from the set \( A \)
    to the set \( B \)
\( f \circ g \) denotes the composition of maps \( g : A \rightarrow B \)
    and \( f : B \rightarrow C \), i.e., for \( p \in A \) we have
    \( (f \circ g)(p) = f[g(p)] \)
\( f[A] \) denotes the image of the set \( A \) under the
    map \( f \), i.e., the set \( \{f(x) | x \in A\} \)

\( C^0 \) the set of \( n \)-times continuously differentiable functions.
    Note that \( C^0 \) means simply continuous, while \( C^1 \)
    means that the first derivative exists and is continuous.
\( C^\infty \) the set of infinitely continuously differentiable
    (i.e., smooth) functions
\( \exists \) there exists; i.e., for all \( u \in \mathbb{R} \), \( \exists \ v \ | \ v + u = 0 \)
\( \forall \) for all; i.e., \( \forall u \in \mathbb{R} \), \( \exists \ v \ | \ v + u = 0 \)
If $f$ is a function $f : M \to N$, $M$ is called the **domain** of $f$, and $N$ is called its **codomain**.

The set of points in $N$ that $M$ gets mapped into is called the **image** of $f$.

For any subset $U \subset N$, the set of elements of $M$ that get mapped to $U$ is called the **preimage** of $U$, or $f^{-1}(U)$.

A map $f : M \to N$ is called **one-to-one** (or **injective**) if each element of $N$ has at most one element of $M$ mapped into it.

A map $f : M \to N$ is called **onto** (or **surjective**) if each element of $N$ has at least one element of $M$ mapped into it.

A map that is both one-to-one and onto is known as **invertible** (or **bijective**). In this case we can define the inverse map $f^{-1} : N \to M$ by $(f^{-1} \circ f)(x) = x$, for any $x \in M$.

(Weierstrass definition): For functions $f : D \to \mathbb{R}$, where $D \subset \mathbb{R}$, $f(x)$ is continuous at $x_0$ if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$.

(General topological definition): If open sets have been defined, then a function $f : X \to Y$ is continuous if and only if the preimage $f^{-1}(V)$, where $V$ is an open subset of $Y$ (which could be the whole set), is always an open subset of $X$.

For the usual definition of open sets on $\mathbb{R}$, the two definitions are equivalent.

If $f : D \to \mathbb{R}^n$, where $D \subset \mathbb{R}^m$, then the definition of continuity is a natural generalization of the $\mathbb{R} \to \mathbb{R}$ definition.

$f$ can be described as a collection of functions $f^i(x^1, x^2, \ldots, x^m)$, where $i = 1, \ldots, n$. $f$ is $C^p$ if each $f^i$ is at least $C^p$ in each of the variables $(x^1, x^2, \ldots, x^m)$.

Suppose that $M$ and $N$ are topological spaces (i.e., spaces on which open sets have been defined). Then if $f : M \to N$ is continuous, one-to-one, and onto, and its inverse is continuous, then $f$ is called a **homeomorphism**, and the spaces $M$ and $N$ are said to be **homeomorphic**. As far as topology is concerned, $M$ and $N$ are then identical. (See Wald. Carroll never uses the word “homeomorphic”.)
Suppose that $M$ and $N$ are manifolds (to be defined shortly). Then if $f : M \to N$ is $C_\infty$, one-to-one, and onto, and its inverse is $C_\infty$, then $f$ is called a **diffeomorphism**, and the spaces $M$ and $N$ are said to be **diffeomorphic**. As far as manifold properties are concerned, $M$ and $N$ are then identical.

An **open ball** is the set of all points $x$ in $\mathbb{R}^n$ such that $|x - y| < r$ for some fixed $y \in \mathbb{R}^n$ and $r \in \mathbb{R}$, where $|x - y|^2 = \sum_i (x_i - y_i)^2$. Note that $|x - y|$ must be less than $r$. The ball does not include its boundary. An **open set** in $\mathbb{R}^n$ is a set constructed from an arbitrary (maybe infinite) union of open balls. Equivalently, a set $V \subset \mathbb{R}^n$ is open if, for any $y \in V$, there is an open ball centered at $y$ that is completely inside $V$.

*A* This entry has been corrected from the version shown in lecture, which mistakenly omitted the requirement that $f$ and $f^{-1}$ must be $C_\infty$.

A $C_\infty$ **atlas** of charts is a collection of charts $\{(U_\alpha, \phi_\alpha)\}$ that satisfies two conditions:

1) The $U_\alpha$ cover $M$, so that any point in $M$ is contained in at least one chart $U_\alpha$.

2) The charts smoothly sew together. Whenever two charts overlap, the map from one coordinate system to the other must be $C_\infty$. In symbols, whenever $U_\alpha \cap U_\beta \neq \emptyset$, the map $\phi_\alpha \circ \phi_\beta^{-1}$ takes points in $\phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$ onto the open set $\phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$. All such maps must be $C_\infty$ where they are defined.
A $C^\infty$ $n$-dimensional manifold (or $n$-manifold for short) is simply a set $M$ along with a maximal atlas, one that contains every possible compatible chart.