8.962 Lecture 14
April 2, 2018

GEODESIC DEVIATION

Geodesic Deviation: Definition

Geodesic deviation is the study of how nearby geodesics evolve relative to each other.

In the absence of gravity and all other forces, the relative velocity of any two geodesics is constant.

Geodesic deviation is a precise formulation of what more generally is referred to as tidal forces.

Three Approaches to Geodesic Deviation

1. Weinberg approach: use coordinates.
2. Carroll approach: stick to covariantly defined tensor identities.
3. "8.962" approach: Use covariantly defined tensor identities, but calculate the scalar quantity $\frac{d^2\ell}{d\tau^2}$, where ℓ is the proper distance between points moving along two nearby trajectories.

Geodesic Deviation, Approach 1: Weinberg's Coordinate Approach

Consider the geodesic equation,

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}(x)\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0 \ ,$$

where τ is an affine parameter, and now consider a nearby trajectory

$$x^\mu(\tau) + \delta x^\mu(\tau) \ .$$

The geodesic equation for the nearby trajectory reads

$$\frac{d^2}{d\tau^2}\left[x^\mu + \delta x^\mu\right] + \Gamma^\mu_{\nu\lambda}(x+\delta x)\frac{d}{d\tau}\left(x^\nu + \delta x^\nu\right)\frac{d}{d\tau}\left(x^\lambda + \delta x^\lambda\right) = 0 \ .$$

Collecting the first order terms,

$$\boxed{\frac{d^2 \delta x^\mu}{d\tau^2} + \frac{\partial \Gamma^\mu_{\nu\lambda}}{\partial \rho}\delta x^\rho \frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} + 2\Gamma^\mu_{\nu\lambda}(x)\frac{dx^\nu}{d\tau}\frac{d\delta x^\lambda}{d\tau} = 0 \ .}$$

$$\boxed{\frac{d^2\delta x^\mu}{d\tau^2} + \frac{\partial\Gamma^\mu_{\nu\lambda}}{\partial\rho}\delta x^\rho \frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} + 2\Gamma^\mu_{\nu\lambda}(x)\frac{dx^\nu}{d\tau}\frac{d\delta x^\lambda}{d\tau} = 0\ .}$$

Virtues of Weinberg approach: simplicity. With a specified coordinate system, the Weinberg approach gives immediately a formula for the relative acceleration $d^2\delta x^\mu/d\tau^2$.

The curvature tensor does not appear in Weinberg's formula. In fact, $d^2\delta x^\mu/d\tau^2$ can be nonzero even in flat space, with $R^\mu{}_{\nu\lambda\sigma} = 0$, if one is using a non-Cartesian coordinate system. This can be good or bad — if you want to know the geodesic deviation in polar coordinates for flat space, this is just what you need. But if you want to understand the connection between curvature and geodesic deviation, which is emphasized by other approaches, then the Weinberg formula does not do it.

Geodesic Deviation, Approach 2: Carroll's Covariant Approach

Consider a family of geodesics, $x^\mu(\sigma,\tau)$. For each σ, $x^\mu(\sigma,\tau)$ is a geodesic in τ.

Define

$$T^\mu \equiv \frac{\partial x^\mu}{\partial \tau} \ , \qquad S^\mu \equiv \frac{\partial x^\mu}{\partial \sigma} \ .$$

T^μ is the tangent vector to the geodesics, which is the 4-velocity.

S^μ is the "geodesic deviation" vector, describing how the geodesic changes with the parameter σ.

Define further

$$V \equiv \nabla_T S \ , \qquad \text{In coords: } V^\mu \equiv T^\rho \nabla_\rho S^\mu = \frac{DS^\mu}{d\tau} \ .$$

V^μ is the "relative velocity of geodesics". Carroll uses these words, in quotes.

$$A^\mu \equiv \nabla_T V \ , \qquad \text{In coords: } A^\mu \equiv T^\sigma \nabla_\sigma \left[T^\rho \nabla_\rho S^\mu\right] \ .$$

A^μ is the "relative acceleration of geodesics".

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A^μ is the "relative acceleration of geodesics".

Note that $\nabla_\rho S^\mu$ is not defined for all ρ, since S^μ is only defined in the space spanned by the 1-parameter family of geodesics. But $T^\rho \nabla_\rho S^\mu$ is well-defined, since T^ρ differentiates within this family of geodesics.

$$\text{Reminder: } T^\mu \equiv \frac{\partial x^\mu}{\partial \tau} \ , \ S^\mu \equiv \frac{\partial x^\mu}{\partial \sigma} \ , \ V \equiv \nabla_T S \ , \ A^\mu \equiv \nabla_T V \ .$$

Carroll writes: "You should take the names with a grain of salt, but these vectors are certainly well-defined."

In particular, the meaning of $A^\mu \equiv T^\sigma \nabla_\sigma \left[T^\rho \nabla_\rho S^\mu\right]$ is subtle, since ∂_σ acts on the affine connections implicit in $\nabla_\rho S^\mu$. So even in locally inertial frame (LIF), where the affine connections vanish but their derivatives don't, A^μ does not reduce to $\partial^2 S^\mu / \partial \tau^2$. So A^μ is equal to the relative acceleration, plus a correction determined by the derivatives of the connection.

Goal: to derive an equation for A^μ.

Important identity:

$$\boxed{\nabla_S T = \nabla_T S \ .}$$

Carroll: "Since S and T are basis vectors adapted to a coordinate system, their commutator vanishes:

$$[S, T] = 0 \ .$$

From

$$[X, Y]^\mu = X^\lambda \nabla_\lambda Y^\mu - Y^\lambda \nabla_\lambda X^\mu \ ,$$

we then have

$$S^\rho \nabla_\rho T^\mu = T^\rho \nabla_\rho S^\mu \ ."$$

Carroll is apparently referring to the (σ, τ) coordinate system of the 2D space of geodesics.

In full detail,

$$
\begin{aligned}
(\nabla_S T)^\mu &= \frac{\partial x^\rho}{\partial \sigma} \nabla_\rho \left(\frac{\partial x^\mu}{\partial \tau} \right) \\
&= \frac{\partial x^\rho}{\partial \sigma} \left[\frac{\partial}{\partial x^\rho} \left(\frac{\partial x^\mu}{\partial \tau} \right) + \Gamma^\mu_{\rho\lambda} \frac{\partial x^\lambda}{\partial \tau} \right] \\
&= \frac{\partial^2 x^\mu}{\partial \sigma \partial \tau} + \Gamma^\mu_{\rho\lambda} \frac{\partial x^\rho}{\partial \sigma} \frac{\partial x^\lambda}{\partial \tau} \ ,
\end{aligned}
$$

and

$$
\begin{aligned}
(\nabla_T S)^\mu &= \frac{\partial x^\rho}{\partial \tau} \nabla_\rho \left(\frac{\partial x^\mu}{\partial \sigma} \right) \\
&= \frac{\partial x^\rho}{\partial \tau} \left[\frac{\partial}{\partial x^\rho} \left(\frac{\partial x^\mu}{\partial \sigma} \right) + \Gamma^\mu_{\rho\lambda} \frac{\partial x^\lambda}{\partial \sigma} \right] \\
&= \frac{\partial^2 x^\mu}{\partial \tau \partial \sigma} + \Gamma^\mu_{\rho\lambda} \frac{\partial x^\rho}{\partial \tau} \frac{\partial x^\lambda}{\partial \sigma} \ .
\end{aligned}
$$

Since ordinary partial derivatives commute, and $\Gamma^\mu_{\rho\lambda} = \Gamma^\mu_{\lambda\rho}$, we have $\nabla_S T = \nabla_T S$.

Then,

$$A^\mu = \frac{D^2 S^\mu}{d\tau^2} = \nabla_T(\nabla_T S)$$
$$= \nabla_T(\nabla_S T)$$
$$= (\nabla_T \nabla_S - \nabla_S \nabla_T)T + \nabla_S \nabla_T T \ .$$

But $\nabla_T T = 0$, by the geodesic equations, so

$$A^\mu = ([\nabla_T, \nabla_S]T)^\mu \ .$$

Recall

$$[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma$$

for any V, so

$$A^\mu = [T^\rho \nabla_\rho, S^\sigma \nabla_\sigma]T^\mu$$
$$= T^\rho \nabla_\rho(S^\sigma \nabla_\sigma T^\mu) - S^\sigma \nabla_\sigma(T^\rho \nabla_\rho T^\mu)$$
$$= (T^\rho \nabla_\rho S^\sigma)\nabla_\sigma T^\mu - (S^\sigma \nabla_\sigma T^\rho)\nabla_\rho T^\mu$$
$$+ T^\rho S^\sigma \nabla_\rho \nabla_\sigma T^\mu - S^\sigma T^\rho \nabla_\sigma \nabla_\rho T^\mu$$
$$= T^\rho S^\sigma [\nabla_\rho, \nabla_\sigma]T^\mu$$
$$= T^\rho S^\sigma R^\mu{}_{\nu\rho\sigma} T^\nu \ .$$

$$
\begin{aligned}
A^\mu &= [T^\rho \nabla_\rho, S^\sigma \nabla_\sigma] T^\mu \\
&= T^\rho \nabla_\rho (S^\sigma \nabla_\sigma T^\mu) - S^\sigma \nabla_\sigma (T^\rho \nabla_\rho T^\mu) \\
&= (T^\rho \nabla_\rho S^\sigma) \nabla_\sigma T^\mu - (S^\sigma \nabla_\sigma T^\rho) \nabla_\rho T^\mu \\
&+ \qquad T^\rho S^\sigma \nabla_\rho \nabla_\sigma T^\mu - S^\sigma T^\rho \nabla_\sigma \nabla_\rho T^\mu \\
&= T^\rho S^\sigma [\nabla_\rho, \nabla_\sigma] T^\mu \\
&= T^\rho S^\sigma R^\mu{}_{\nu\rho\sigma} T^\nu \ .
\end{aligned}
$$

So finally,

$$
\boxed{A^\mu = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma \ .}
$$