8.962 Lecture 16
April 9, 2018

SECOND BIANCHI
IDENTITY

Application to $R^\rho_{\sigma\mu\nu}$

Recall: $[\nabla_\mu, \nabla_\nu] V^\sigma = R^\rho_{\sigma\mu\nu} V^\sigma$.

Then: $[\nabla_\mu, \nabla_\nu] V_\rho = R^\rho_{\rho\mu\nu} V^\sigma = -R^\rho_{\sigma\mu\nu} V^\sigma = -R^\rho_{\sigma\nu\mu} V_\sigma$.

Now consider: $[\nabla_\mu, \nabla_\nu] V_\rho W_\tau$.

Claim: $[\nabla_\mu, \nabla_\nu] V_\rho W_\tau = -R^\rho_{\rho\mu\nu} V_\rho W_\tau - R^\rho_{\rho\nu\mu} V_\nu W_\tau$.

Explanation: When $[\nabla_\mu, \nabla_\nu] V_\rho W_\tau$ is expanded, using the Leibnitz rule, the terms shown above are generated when both $\nabla$’s act on $V_\rho$ or $W_\tau$. When one $\nabla$ acts on each vector, the terms cancel. For example,

$[\nabla_\mu, \nabla_\nu] V_\rho W_\tau = \nabla_\rho [\nabla_\mu, \nabla_\nu] V_\tau - \nabla_\nu [\nabla_\mu, \nabla_\nu] V_\rho W_\tau$,

which contains

$(\nabla_\rho V_\nu)(\nabla_\mu W_\tau) - (\nabla_\mu V_\rho)(\nabla_\nu W_\tau) = 0$.

But tensors of the form $V_\rho W_\tau$ span the space of all (0,2) tensors $T_{\rho\tau}$, so

$[\nabla_\mu, \nabla_\nu] T_{\rho\tau} = -R^\rho_{\rho\mu\nu} T_{\sigma\tau} - R^\rho_{\rho\nu\mu} T_{\tau\sigma}$.

Second Bianchi Identity

Basic idea:

$R^\rho_{\sigma\mu\nu}$ is a commutator.

Commutators obey the Jacobi identity:


Proof of Jacobi identity: Just write it out. For example, the term $ABC$ occurs positively in first Jacobi term, and negatively in last Jacobi term.

So, the Jacobi identity should lead to an identity for the Riemann tensor.

What is $A$? Ans: $\nabla_\lambda$.

Applying the Jacobi Identity

$[\nabla_\lambda, [\nabla_\mu, \nabla_\nu]] V_\sigma + \text{cyclic}(\lambda, \mu, \nu) = 0$,

where $+\text{cyclic}(\lambda, \mu, \nu)$ means to add the two other cyclic permutations of the first term.

Expanding the outer commutator,

$\nabla_\lambda ([\nabla_\mu, \nabla_\nu] V_\rho) - [\nabla_\mu, \nabla_\nu] \nabla_\lambda V_\rho + \text{cyclic}(\lambda, \mu, \nu) = 0$.

Using $[\nabla_\mu, \nabla_\nu] V_\rho = -R^\rho_{\rho\mu\nu} V_\sigma$, etc.,

$-\nabla_\lambda (R^\rho_{\rho\mu\nu} V_\rho) + R^\rho_{\lambda\mu\nu} \nabla_\rho V_\sigma + R^\rho_{\rho\mu\nu} \nabla_\lambda V_\rho + \text{cyclic}(\lambda, \mu, \nu) = 0$.
Using $[\nabla_\mu, \nabla_\nu] V_\rho = -R^\sigma_{\mu\rho\nu} V_\sigma$, etc.,

$$-\nabla_\lambda (R^\rho_{\sigma\mu\nu} V_\rho) + R^\rho_{\lambda\mu\nu} \nabla_\mu V_\sigma + R^\rho_{\sigma\mu\nu} \nabla_\lambda V_\rho + \text{cyclic}(\lambda, \mu, \nu) = 0 .$$

Expanding the 1st term via Leibnitz, the 3rd term is canceled, leaving

$$\nabla_\lambda (R^\rho_{\sigma\mu\nu} V_\rho) - R^\rho_{\lambda\mu\nu} \nabla_\mu V_\sigma + \text{cyclic}(\lambda, \mu, \nu) = 0 .$$

Now notice that, by the first Bianchi identity, the 2nd term vanishes when cyclically summed. So, finally,

$$\nabla_\lambda R^\rho_{\sigma\mu\nu} + \text{cyclic}(\lambda, \mu, \nu) = 0 .$$