The TOV equations in the $r = 0$ limit

I've gotten a few question about the numerical integration of the TOV equations of stellar structure, in particular wondering how to handle the fact that the equations seem to be ill behaved at $r = 0$. The issue here is due to the pressure gradient equation:

$$\frac{dP}{dr} = -\frac{G(\rho + P)[m(r) + 4\pi r^3 P]}{r[r - 2Gm(r)]} \; .$$

We know that as $r \to 0$, $m(r) \to 0$, $P \to P_c$ and $\rho \to \rho_c$ (the values at the center). Hence,

$$\left.\frac{dP}{dr}\right|_{r\to 0} \to -G(\rho_c + P_c)\frac{4\pi r^3 P_c}{r^2} \to 0 \; .$$

In other words, dP/dr is perfectly well behaved as $r \to 0$. Unfortunately, it's well behaved *in the limit*. Many numerical integrators just see 0/0 and proceed to pout.

The solution is to use the limiting behavior to do the first bit analytically, and then start from there. Pick some $r_0 \ll GM$, where M is the total mass of the star. You don't know what this is, but it will be something of order kilometers. Picking $r_0 \sim$ centimeters should be good. Over this lengthscale, you can use the fact that $dP/dr \approx 0$ (with the approximation becoming exact at $r = 0$) to argue that the pressure is approximately constant and hence the density is approximately constant. Then, you know

$$m(r_0) \approx \frac{4}{3}\pi r_0^3 \rho_c \; .$$

This, plus $P(r_0) \approx P_c$, $\rho(r_0) \approx \rho_c$ give us a nice, nonsingular solution which we can use to start the integral at $r = r_0$. (The approximations become more and more accurate the smaller we make r_0.)

With this, you should be able to integrate pretty well.