Some notes on Pset 6 problem 1

Today in recitation, the issue came up of whether for the purpose of breaking up the vacuum Einstein equation into "constraint" and "evolution" equations it mattered whether we place our indices. I said no, then was convinced that the answer is yes. Later in the day, I then changed my mind; but I have now changed it back.

For the purpose for which the problem is intended, *the placement of the indices DOES matter*. The key point is the one that William Throwe made during class: By choosing a specific way to split "spacetime" into "space" and "time," lose the coordinate invariance of the original tensor equation we started out with.

The intent of the problem is to show that certain components of the Einstein tensor do not have any second derivatives with respect to our time coordinate, and can thus be considered to be constraints; the remaining components *do* have second derivatives and can thus be consider evolution equations. What I should have specified is that the operator is with respect to the *coordinate* $t \equiv x^0$; as such, the more natural way to phrase the form of the Bianchi identity you want to use is

$$\nabla_\alpha G^{\alpha\beta} = 0\ .$$

From this starting point you should be able to prove that

$$\begin{aligned} &G^{ij} \text{ contains terms with } \partial_t^2 \\ &G^{0j} \text{ contains terms with } \partial_t \\ &G^{00} \text{ contains no time derivatives.} \end{aligned}$$

I won't say any more about this because the solution to this is the point of the assigned problem.

Does it follow that a similar pattern holds for G_{ij}, G_{0j}, G_{00}? A few moments of thought can convince you that the answer is certainly *no*. Consider, for example G_{00}. It can be constructed from the upstairs-upstairs components as follows:

$$\begin{aligned} G_{00} &= g_{0\mu} g_{0\nu} G^{\mu\nu} \\ &= (g_{00})^2 G^{00} + 2 g_{00} g_{0j} G^{0j} + g_{ik} g_{jl} G^{kl}\ . \end{aligned}$$

From this we can see that G_{00} *must* contain terms with ∂_t^2!

My advice for doing the problem is to shift all the indices into the upstairs position and proceed that way. In the last week of the class, we will see that this problem is a somewhat clumsy example of the way that the "real" constraint and evolution equations are written down from the Einstein equations. In that analysis, we will proceed by

- Choosing a global time coordinate t
- Defining a *spacetime vector* $\vec{n}$ which is normal to hypersurfaces of constant t
- Making constraint and evolution equations by projecting $G_{\alpha\beta} = 8\pi G T_{\alpha\beta}$ along and orthogonal to the normal $\vec{n}$.

Because this construction is done using real tensors and real tensor algebra (rather than just picking components in some coordinates system), the flakiness of our split into "space" and "time" does not enter.