Conservation of the stress-energy tensor

There is a short piece of material that I did not have time to cover in my 12 Feb 2009 lecture which is important background to Problem Set #2. Recall that $\partial_\alpha N^\alpha = 0$ expresses conservation of number density in a covariant, geometric manner. We want to formulate conservation of energy and momentum in a similar way. The only way to do this in a covariant, geometric manner is to combine the two laws into

$$\boxed{\boxed{\partial_\alpha T^{\alpha\beta} = 0\ .}}$$

To see that this corresponds to conservation of energy and conservation of momentum, we need to look at it in some observer's particular reference frame. Let's choose an observer with coordinates (t, x, y, z), and begin with the timelike component of this more general conservation law:

$$\partial_\alpha T^{\alpha 0} = 0\ ;$$

or,

$$\frac{\partial T^{00}}{\partial t} = -\frac{\partial T^{i0}}{\partial x^i}\ .$$

In words, the rate of change of energy density T^{00} measured by our observer is equal to the (minus) divergence of the energy flux. Exactly the form we expect of energy conservation on intuitive grounds! Integrating both sides over a 3-volume and invoking Gauss, we come to an integral formulation of energy conservation:

$$\frac{\partial}{\partial t}\int_{V^3} T^{00} dV^3 = -\int_{\partial V^3} T^{0i} d\Sigma_i\ .$$

In words, the rate of change of energy in V^3 is balanced by the flux of energy across the boundary ∂V^3.

Likewise, the spatial component of our general law gives us momentum conservation:

$$\partial_\alpha T^{\alpha j} = 0 \quad \longrightarrow \quad \frac{\partial T^{0j}}{\partial t} = -\frac{\partial T^{ij}}{\partial x^i}\ ,$$

$$\frac{\partial}{\partial t}\int_{V^3} T^{0j} dV^3 = -\int_{\partial V^3} T^{ij} d\Sigma_i\ ;$$

the rate of change of momentum in V^3 is balanced by the momentum flux across the boundary ∂V^3.