Massachusetts Institute of Technology
Department of Physics
8.962 Spring 2009

Problem Set 2

Post date: Thursday, February 12th
Due date: Thursday, February 19th

1. [5 pts] Show that the number density of dust measured by an observer whose 4-velocity is $\vec{U}$ is given by $n = -\vec{N} \cdot \vec{U}$, where $\vec{N}$ is the matter current 4-vector.

2. [5 pts] Take the limit of the continuity equation for $|\mathbf{v}| \ll 1$ to get $\partial n/\partial t + \partial(nv^i)/\partial x^i = 0$.

   Note: Discussion following lecture on 12 Feb 2009 made me realize that as worded, it may be somewhat ambiguous to what order this should be expanded. You may find it helpful to reinsert factors of c, so that $\vec{u} \doteq (\gamma c, \gamma v_i)$, and $\partial_\alpha \doteq ((1/c)\partial_t, \partial_i)$. The limit we are considering is the one in which there are no $1/c$, $1/c^2$, etc terms; it is essentially the $c \to \infty$ limit. You may find it interesting to consider terms beyond this limit.

3. In an inertial frame $\mathcal{O}$, calculate the components of the stress-energy tensors of the following systems:

   (a) [5 pts] A group of particles all moving with the same 3-velocity $\mathbf{v} = \beta \vec{e}_x$ as seen in $\mathcal{O}$. Let the rest-mass density of these particles be ρ_0, as measured in their own rest frame. Assume a sufficiently high density of particles to enable treating them as a continuum.

   (b) [10 pts] A ring of N similar particles of rest mass m rotating counter-clockwise in the $x - y$ plane about the origin of $\mathcal{O}$, at a radius a from this point, with an angular velocity ω. The ring is a torus whose circular cross section is of radius $\delta a \ll a$. Within this cross section, the particles are uniformly distributed with a high enough density for the continuum approximation to apply. Do not include the stress-energy of whatever forces keep them in orbit. Part of this calculation will relate ρ_0 of part (a) to N, a, δa, and ω.

   (c) [5 pts] Two such rings of particles, one rotating clockwise and the other counter-clockwise, at the same radius a. The particles do not collide or otherwise interact in any way.

4. (Notes I just posted on the conservation of the stress-energy tensor may prove to be useful background for this problem.) Use the identity $\partial_\nu T^{\mu\nu} = 0$ to prove the following results for a bounded system (i.e., a system for which $T^{\mu\nu} = 0$ beyond some bounded region of space):

   (a) [3 pts] In some inertial frame, $\partial_t \int T^{0\alpha} d^3x = 0$. This expresses conservation of energy and momentum.

   (b) [6 pts] $\partial_t^2 \int T^{00} x^i x^j d^3x = 2 \int T^{ij} d^3x$. This result is a version of the virial theorem; it will come in quite handy when we derive the quadrupole formula for gravitational radiation.

   (c) [6 pts] $\partial_t^2 \int T^{00} (x^i x_i)^2 d^3x = 4 \int T^i{}_i x^j x_j d^3x + 8 \int T^{ij} x_i x_j d^3x$. No pithy wisdom for this one.

5. The vector potential $\vec{A} \doteq (A^0, \mathbf{A})$ generates the electromagnetic field tensor via

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ .$$

(a) [5 pts] Show that the electric and magnetic fields in a specific Lorentz frame are given by

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A} \\ \mathbf{E} &= -\frac{\partial \mathbf{A}}{\partial t} - \nabla A^0 \ . \end{aligned}$$

Here, ∇ is taken to be the normal gradient operator in Euclidean space.

(b) [5 pts] Show that Maxwell's equations hold if and only if

$$\partial_\mu \partial^\mu A^\alpha - \partial^\alpha \partial_\mu A^\mu = -4\pi J^\alpha \ .$$

(c) [5 pts] Show that a gauge transformation of the form

$$A_\mu^{\text{new}} = A_\mu^{\text{old}} + \partial_\mu \phi$$

leaves the field tensor unchanged.

(d) [5 pts] Show that one can adjust the gauge so that

$$\partial_\mu A^\mu = 0 \ .$$

Show that Maxwell's equations take on a particularly simple form with this gauge choice. Use the operator $\Box \equiv \partial_\mu \partial^\mu$ to simplify your result.

6. A coordinate system for a uniformly accelerating observer

(Note: A fairly large amount of background definitions prior to formulating the question itself. You might want to just scan this material initially, then come back to it as you work through the parts of this problem.)

A former 8.962 student is now an astronaut. She moves through space with acceleration g in the x direction. In other words, her 4-acceleration $\vec{a} = d\vec{u}/d\tau$ (where τ is time as measured on the astronaut's own clock) only has spatial components in the x direction, and is normalized such that

$$\sqrt{\vec{a} \cdot \vec{a}} = g \; .$$

This astronaut assigns coordinates $(\bar{t}, \bar{x}, \bar{y}, \bar{z})$ as follows:

First, she defines spatial coordinates to be $(\bar{x}, \bar{y}, \bar{z})$, and sets the time coordinate $\bar{t}$ to be her own proper time. She defines her position to be $(\bar{x} = g^{-1}, \bar{y} = 0, \bar{z} = 0)$ (not a unique choice, but a convenient one). Note that she remains *fixed* with respect to these coordinates — that's the point of coordinates for an accelerated observer!

Second, at $\bar{t} = 0$, the astronaut chooses $(\bar{t}, \bar{x}, \bar{y}, \bar{z})$ to coincide with the Euclidean coordinates (t, x, y, z) of the inertial reference frame that momentarily coincides with her motion. In other words, though the astronaut is not inertial, there is an inertial frame that, at $\bar{t} = 0$, is momentarily at rest with respect to her. This is the frame used to assign $(\bar{x}, \bar{y}, \bar{z})$ at $\bar{t} = 0$. The clocks of that frame are set such that they are synchronized with her clock at that moment.

Observers who remain at fixed values of the spatial coordinates $(\bar{x}, \bar{y}, \bar{z})$ are called coordinate-stationary observers (CSOs). Note that the CSOs are also accelerated observers, though not necessarily accelerating at the same rate as the astronaut. The astronaut requires the CSO worldlines to be orthogonal to hypersurfaces $\bar{t} =$ constant. She also requires that for each $\bar{t}$ there exists some inertial frame, momentarily at rest with respect to the astronaut, in which all events with $\bar{t} =$ constant are simultaneous. The accelerated motion of the astronaut can thus be described as movement through a sequence of inertial frames which momentarily coincide with her motion.

It is easy to see that $\bar{y} = y$ and $\bar{z} = z$; henceforth we drop these coordinates from the problem.

(a) [5 pts] What is the 4-velocity $\vec{u}$ of the astronaut, as a function of $\bar{t}$, as measured by CSOs in the initial inertial frame [the frame that uses coordinates (t, x, y, z)]? (Hint: by considering the conditions on $\vec{u} \cdot \vec{u}$, $\vec{u} \cdot \vec{a}$, and $\vec{a} \cdot \vec{a}$, you should be able to find simple forms for u^t and u^x.) After you have worked out $\vec{u}$, compute $\vec{a}$.

(b) [5 pts] Integrate up this 4-velocity to find the position $[T(\bar{t}), X(\bar{t})]$ of the astronaut in the coordinates (t, x). Recall that at $t = \bar{t} = 0$, $X = \bar{x} = g^{-1}$. Sketch the astronaut's worldline on a spacetime diagram in the coordinates (t, x). You will return to and augment this sketch over the course of this problem, so you may want to do this on a separate piece of paper (and/or clean it up before handing in your assignment).

(c) [5 pts] Find basis vectors $\vec{e}_{\bar{t}}$ and $\vec{e}_{\bar{x}}$ describing the momentarily inertial coordinate system at some time $\bar{t}$. **Hint**: In lecture, I noted that a body with a 4-velocity $\vec{u}$ has a very natural basis for the timelike direction. Now find an orthogonal vector in this problem; it will serve as $\vec{e}_{\bar{x}}$ (possibly modulo normalization). Add these vectors to the sketch of your worldline.

We now "promote" $\bar{t}$ to a coordinate (i.e., give it meaning not just on the astronaut's worldline, but everywhere in spacetime) by requiring that $\bar{t} =$ constant be a surface of constant time in the Lorentz frame in which the astronaut is instantaneously at rest.

(d) [5 pts] By noting that this "surface" must be parallel to $\vec{e}_{\bar{x}}$ and that it must pass through the point $[T(\bar{t}), X(\bar{t})]$, show that it is defined by the line

$$x = t \coth g\bar{t} \ .$$

In other words, it's just a straight line going through the origin with slope $\coth g\bar{t}$.

We've now defined the time coordinate $\bar{t}$ that the astronaut uses to label spacetime. Next, we need to come up with a way to set her spatial coordinates $\bar{x}$.

(e) [5 pts] Recalling that the CSOs must themselves be accelerated observers, argue that their worldlines are hyperbolae[1], and thus that a CSO's position in (t, x) must take the form

$$t = \frac{A}{g} \sinh g\bar{t} \ ,$$
$$x = \frac{A}{g} \cosh g\bar{t} \ .$$

From the initial conditions, find A.

(f) [10 pts] Show that the line element $ds^2 = d\vec{x} \cdot d\vec{x}$ in the new coordinates takes the form

$$ds^2 = -dt^2 + dx^2 = -(g\bar{x})^2 d\bar{t}^2 + d\bar{x}^2.$$

This is known as the *Rindler metric*[2]. As this exercise illustrates, it is just the flat spacetime of special relativity; but, it is expressed in coordinates that introduce some features that are very important in general relativity. We will return to this spacetime in later exercises.

[1] If your sketch of the astronaut's worldline isn't a hyperbola, you might want to revise your solution.

[2] Rindler often appears in textbooks with $(1 + g\bar{x})^2$ rather than $(g\bar{x})^2$. The difference amounts to an uninteresting shift of the origin.