

MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
DEPARTMENT OF PHYSICS  
8.962 SPRING 2008

PROBLEM SET 4

Post date: Thursday, March 6th

Due date: Thursday, March 13th

1. [10 pts] Connection in Rindler spacetime

Recall that “Rindler spacetime”, which describes flat spacetime as observed by an accelerated observer, that we derived on Pset 2:

$$ds^2 = -(1 + g\bar{x})^2 d\bar{t}^2 + d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2 \quad (1)$$

(I’ve shifted the origin slightly, so that  $g\bar{x} \rightarrow 1 + g\bar{x}$ .) Compute all non-zero Christoffel symbols for this spacetime. (Carroll problem 3.3 will help you quite a bit here.)

2. Relativistic Euler equation

(a) [8 pts] Recall the stress-energy tensor for a perfect fluid,  $T_{\alpha\beta} = \rho u_\alpha u_\beta + P h_{\alpha\beta}$ , where  $h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$  (where  $u^\alpha$  describes the 4-velocity of an element of the fluid). Use local energy momentum conservation,  $\nabla_\alpha T^{\alpha\beta} = 0$ , to derive the relativistic Euler equation,

$$(\rho + P)u^\alpha \nabla_\alpha u^\beta = -h^{\alpha\beta} \nabla_\alpha P. \quad (2)$$

**Comment:** If you just evaluate  $\nabla_\alpha T^{\alpha\beta} = 0$ , you will have a hard time getting a useful answer. You will get something that, technically is *an* Euler equation, but it won’t be one that reduces naturally to a form that is useful for most calculations. In particular, you won’t be able to obtain the non-relativistic limit in part (b).

A more useful form is obtained by separately equating to zero the components of  $\nabla_\alpha T^{\alpha\beta}$  parallel to and orthogonal to the fluid’s 4-velocity,  $u^\alpha$ . Define  $j^\beta = \nabla_\alpha T^{\alpha\beta}$ . Then, the equations

$$\begin{aligned} j^\beta u_\beta &= 0 && \text{(Component parallel to } \vec{u}) \\ j^\beta h^\gamma{}_\beta &= 0 && \text{(Component orthogonal to } \vec{u}) \end{aligned}$$

give us useful information. The second equation in particular should provide us with a useful Euler equation. (Any idea what the first equation tells us?)

(b) [4 pts] For a nonrelativistic fluid ( $\rho \gg P$ ,  $v^t \gg v^i$ ) and in a cartesian basis, show that this equation reduces to the Euler equation,

$$\frac{\partial v_i}{\partial t} + v_k \partial_k v_i = -\frac{1}{\rho} \partial_i P. \quad (3)$$

( $i, k$  are spatial indices running from 1 to 3.) What extra terms are present if the connection is non-zero (e.g., spherical coordinates)?

(c) [6 pts] Apply the relativistic Euler equation to Rindler spacetime for hydrostatic equilibrium. Hydrostatic equilibrium means that the fluid is at rest in the  $\bar{x}$  coordinates, i.e.  $u^{\bar{x}} = 0$ . Suppose that the equation of state (relation between pressure and

density) is  $P = w\rho$  where  $w$  is a positive constant. Find the general solution  $\rho(\bar{x})$  with  $\rho(0) = \rho_0$ .

(d) [6 pts] Suppose now instead that  $w = w_0/(1 + g\bar{x})$  where  $w_0$  is a constant. Show that the solution is  $\rho(\bar{x}) = \rho_0 \exp(-\bar{x}/L)$ . Find  $L$ , the density scale height, in terms of  $g$  and  $w_0$ . Convert to “normal” units by inserting appropriate factors of  $c$  —  $L$  should be a length.

(e) [6 pts] Compare your solution to the density profile of a nonrelativistic, plane-parallel, isothermal atmosphere (for which  $P = \rho kT/\mu$ , where  $T$  is temperature and  $\mu$  is the mean molecular weight) in a constant gravitational field. [Use the nonrelativistic Euler equation with gravity: add a term  $-\partial_i\Phi = g_i$ , where  $\Phi$  is Newtonian gravitational potential and  $g_i$  is Newtonian gravitational acceleration, to the right hand side of Eq. (3).] Why does hydrostatic equilibrium in Rindler spacetime — where there is no gravity — give such similar results to hydrostatic equilibrium in a gravitational field?

### 3. [20 pts] Spherical hydrostatic equilibrium

As we shall derive later in the course, the line element for a spherically symmetric static spacetime may be written

$$ds^2 = -e^{2\Phi(r)} dt^2 + \left[ 1 - \frac{2GM(r)}{r} \right]^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where  $\Phi(r)$  and  $M(r)$  are some given functions. In hydrostatic equilibrium, the spatial components of the fluid’s 4-velocity are all zero:  $u^i = 0$  for  $i \in [r, \theta, \phi]$ . Using the relativistic Euler equation, show that in hydrostatic equilibrium  $P = P(r)$  with

$$\frac{\partial p}{\partial r} = -(\rho + P) \frac{\partial \Phi}{\partial r}.$$

(Hint: Don’t forget to enforce  $\vec{u} \cdot \vec{u} = g_{\alpha\beta} u^\alpha u^\beta = -1!$ )

### 4. [20 pts] Converting from non-affine to affine parameterization

Suppose  $v^\alpha = dx^\alpha/d\lambda^*$  obeys the geodesic equation in the form

$$\frac{Dv^\alpha}{d\lambda^*} = \kappa(\lambda^*)v^\alpha.$$

Clearly  $\lambda^*$  is not an affine parameter.

Show that  $u^\alpha = dx^\alpha/d\lambda$  obeys the geodesic equation in the form

$$\frac{Du^\alpha}{d\lambda} = 0$$

provided that

$$\frac{d\lambda}{d\lambda^*} = \exp \left[ \int \kappa(\lambda^*) d\lambda^* \right].$$

5. [20 pts] Conserved quantities with charge

A particle with electric charge  $e$  moves with 4-velocity  $u^\alpha$  in a spacetime with metric  $g_{\alpha\beta}$  in the presence of a vector potential  $A_\mu$ . The equation describing this particle's motion can be written

$$u^\beta \nabla_\beta u_\alpha = e F_{\alpha\beta} u^\beta ,$$

where

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha .$$

The spacetime admits a Killing vector field  $\xi^\alpha$  such that

$$\begin{aligned} \mathcal{L}_\xi g_{\alpha\beta} &= 0 , \\ \mathcal{L}_\xi A_\alpha &= 0 . \end{aligned}$$

Show that the quantity  $(u_\alpha + eA_\alpha)\xi^\alpha$  is constant along the worldline of the particle.