

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS
8.962 SPRING 2009

PROBLEM SET 4

Post date: Thursday, March 5th

Due date: Thursday, March 12th

1. [10 pts] Connection in Rindler spacetime

Recall the “Rindler spacetime”, which describes flat spacetime as observed by an accelerated observer, that we derived on Pset 2:

$$ds^2 = -(1 + g\bar{x})^2 d\bar{t}^2 + d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2 \quad (1)$$

(I’ve shifted the origin slightly, so that $g\bar{x} \rightarrow 1 + g\bar{x}$.) Compute all non-zero Christoffel symbols for this spacetime. (Carroll problem 3.3 will help you quite a bit here.)

2. Relativistic Euler equation

(a) [8 pts] Recall the stress-energy tensor for a perfect fluid, $T_{\alpha\beta} = \rho u_\alpha u_\beta + P h_{\alpha\beta}$, where $h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$ (where u^α describes the 4-velocity of an element of the fluid; note that $h_{\alpha\beta}$ is the tensor which projects orthogonal to u^α [cf. Pset 1]). Use local energy momentum conservation, $\nabla_\alpha T^{\alpha\beta} = 0$, to derive the relativistic Euler equation,

$$(\rho + P)u^\alpha \nabla_\alpha u^\beta = -h^{\alpha\beta} \nabla_\alpha P . \quad (2)$$

Comment: If you just evaluate $\nabla_\alpha T^{\alpha\beta} = 0$, you will have a hard time getting a useful answer. You will get something that, technically is *an* Euler equation, but it won’t be one that reduces naturally to a form that is useful for most calculations. In particular, you won’t be able to obtain the non-relativistic limit in part (b).

A more useful form is obtained by separately equating to zero the components of $\nabla_\alpha T^{\alpha\beta}$ parallel to and orthogonal to the fluid’s 4-velocity, u^α . Define $j^\beta = \nabla_\alpha T^{\alpha\beta}$. Then, the equations

$$\begin{aligned} j^\beta u_\beta &= 0 && \text{(Component parallel to } \vec{u}) \\ j^\beta h^\gamma_\beta &= 0 && \text{(Component orthogonal to } \vec{u}) \end{aligned}$$

give us useful information. The second equation in particular should provide us with a useful Euler equation. (Any idea what the first equation tells us?)

(b) [4 pts] For a nonrelativistic fluid ($\rho \gg P$, $v^t \gg v^i$) and in a Cartesian basis, show that this equation reduces to the Euler equation,

$$\frac{\partial v_i}{\partial t} + v_k \partial_k v_i = -\frac{1}{\rho} \partial_i P . \quad (3)$$

(i, k are spatial indices running from 1 to 3. We are being a bit sloppy with the summation convention since the spatial metric is just the Kronecker delta, hence $x^i = x_i$.) What extra terms are present if the connection is non-zero (e.g., spherical coordinates)?

(c) [6 pts] Apply the relativistic Euler equation to Rindler spacetime for hydrostatic equilibrium. Hydrostatic equilibrium means that the fluid is at rest in the \bar{x} coordinates, i.e. $u^{\bar{x}} = 0$. Suppose that the equation of state (relation between pressure and density) is $P = w\rho$ where w is a positive constant. Find the general solution $\rho(\bar{x})$ with $\rho(0) = \rho_0$.

(d) [6 pts] Suppose now instead that $w = w_0/(1 + g\bar{x})$ where w_0 is a constant. Show that the solution is $\rho(\bar{x}) = \rho_0 \exp(-\bar{x}/L)$. Find L , the density scale height, in terms of g and w_0 . Convert to “normal” units by inserting appropriate factors of c — L should be a length.

(e) [6 pts] Compare your solution to the density profile of a nonrelativistic, plane-parallel, isothermal atmosphere (for which $P = \rho kT/\mu$, where T is temperature and μ is the mean molecular weight) in a constant gravitational field. [Use the nonrelativistic Euler equation with gravity: add a term $-\partial_i\Phi = g_i$, where Φ is Newtonian gravitational potential and g_i is Newtonian gravitational acceleration, to the right hand side of Eq. (3).] Why does hydrostatic equilibrium in Rindler spacetime — where there is no gravity — give such similar results to hydrostatic equilibrium in a gravitational field?

3. [20 pts] Spherical hydrostatic equilibrium

As we shall derive later in the course, the line element for a spherically symmetric static spacetime may be written

$$ds^2 = -e^{2\Phi(r)} dt^2 + \left[1 - \frac{2GM(r)}{r}\right]^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where $\Phi(r)$ and $M(r)$ are some given functions. In hydrostatic equilibrium, the spatial components of the fluid’s 4-velocity are all zero: $u^i = 0$ for $i \in [r, \theta, \phi]$. Using the relativistic Euler equation, show that in hydrostatic equilibrium $P = P(r)$ with

$$\frac{\partial p}{\partial r} = -(\rho + P) \frac{\partial \Phi}{\partial r}.$$

(Hint: Don’t forget to enforce $\vec{u} \cdot \vec{u} = g_{\alpha\beta} u^\alpha u^\beta = -1!$)

4. [20 pts] Converting from non-affine to affine parameterization

Suppose $v^\alpha = dx^\alpha/d\lambda^*$ obeys the geodesic equation in the form

$$\frac{Dv^\alpha}{d\lambda^*} = \kappa(\lambda^*)v^\alpha.$$

Clearly λ^* is not an affine parameter.

Show that $u^\alpha = dx^\alpha/d\lambda$ obeys the geodesic equation in the form

$$\frac{Du^\alpha}{d\lambda} = 0$$

provided that

$$\frac{d\lambda}{d\lambda^*} = \exp \left[\int \kappa(\lambda^*) d\lambda^* \right].$$

5. [20 pts] Conserved quantities with charge

A particle with electric charge e moves with 4-velocity u^α in a spacetime with metric $g_{\alpha\beta}$ in the presence of a vector potential A_μ . The equation describing this particle's motion can be written

$$u^\beta \nabla_\beta u_\alpha = e F_{\alpha\beta} u^\beta ,$$

where

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha .$$

The spacetime admits a Killing vector field ξ^α such that

$$\begin{aligned} \mathcal{L}_\xi g_{\alpha\beta} &= 0 , \\ \mathcal{L}_\xi A_\alpha &= 0 . \end{aligned}$$

Show that the quantity $(u_\alpha + eA_\alpha)\xi^\alpha$ is constant along the worldline of the particle.