8.962 Pset 4

Problem 1

We define the coordinates (no bars, because they're hard to type) we will use for spacetime:

```
In[11]:= coord = {t, x, y, z}

Out[11]= {t, x, y, z}
```

And the metric ($g_{\mu\nu}$). We will use a for the scalar acceleration, not g, because we want to use g for the metric:

```
In[12]:= g = DiagonalMatrix[{-(1 + a x)^2, 1, 1, 1}]
```

$$\text{Out[12]= } \{\{-(1 + a\,x)^2, 0, 0, 0\}, \{0, 1, 0, 0\}, \{0, 0, 1, 0\}, \{0, 0, 0, 1\}\}$$

```
In[13]:= gInv = Inverse[g] // FullSimplify
```

$$\text{Out[13]= } \left\{\left\{-\frac{1}{(1 + a\,x)^2}, 0, 0, 0\right\}, \{0, 1, 0, 0\}, \{0, 0, 1, 0\}, \{0, 0, 0, 1\}\right\}$$

Here are the connection coefficients, $(\Gamma^{\alpha})_{\beta\gamma}$:

```
In[17]:= MatrixForm[Γ = Table[Sum[1 / 2 gInv[[α, κ]]
              (D[g[[γ, κ]], coord[[β]]] + D[g[[β, κ]], coord[[γ]]] - D[g[[β, γ]], coord[[κ]]]),
            {κ, 1, 4}], {α, 1, 4}, {β, 1, 4}, {γ, 1, 4}] // FullSimplify]
```

Out[17]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} 0 \\ \frac{a}{1+a\,x} \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \frac{a}{1+a\,x} \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} a\,(1+a\,x) \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix}$$

Problem 3

Here are the coordinates.

```
In[1]:= coord = {t, r, θ, ϕ}
```

Out[1]= $\{t, r, \theta, \phi\}$

The metric

```
In[2]:= g = DiagonalMatrix[{-Exp[2 Φ[r]], 1 / (1 - 2 G M[r] / r), r^2, r^2 Sin[θ]^2}]
```

Out[2]= $\left\{\{-e^{2\Phi[r]}, 0, 0, 0\}, \left\{0, \frac{1}{1-\frac{2 G M[r]}{r}}, 0, 0\right\}, \{0, 0, r^2, 0\}, \{0, 0, 0, r^2 \mathrm{Sin}[\theta]^2\}\right\}$

The inverse metric.

```
In[3]:= gInv = FullSimplify[Inverse[g]]
```

Out[3]= $\left\{\{-e^{-2\Phi[r]}, 0, 0, 0\}, \left\{0, 1-\frac{2 G M[r]}{r}, 0, 0\right\}, \left\{0, 0, \frac{1}{r^2}, 0\right\}, \left\{0, 0, 0, \frac{\mathrm{Csc}[\theta]^2}{r^2}\right\}\right\}$

And the Christoffel symbols.

```
In[4]:= MatrixForm[Γ = Table[Sum[1 / 2 gInv[[α, κ]]
            (D[g[[β, κ]], coord[[γ]]] + D[g[[γ, κ]], coord[[β]]] - D[g[[β, γ]], coord[[κ]]]),
          {κ, 1, 4}], {α, 1, 4}, {β, 1, 4}, {γ, 1, 4}] // FullSimplify]
```

Out[4]//MatrixForm=

$$\begin{pmatrix}
\begin{pmatrix} 0 \\ \Phi'[r] \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \Phi'[r] \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} \frac{e^{2\Phi[r]}(r-2 G M[r])\Phi'[r]}{r} \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ \frac{-G M[r]+G r M'[r]}{r(r-2 G M[r])} \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ -r+2 G M[r] \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ -(r-2 G M[r])\mathrm{Sin}[\theta]^2 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ \frac{1}{r} \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ \frac{1}{r} \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta] \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{r} \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathrm{Cot}[\theta] \end{pmatrix} & \begin{pmatrix} 0 \\ \frac{1}{r} \\ \mathrm{Cot}[\theta] \\ 0 \end{pmatrix}
\end{pmatrix}$$

And we see that the only non-zero Γ of the form $(\Gamma^i)_{t\,t}$ is $(\Gamma^r)_{t\,t}$