

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS
8.962 SPRING 2008

PROBLEM SET 5

Post date: Thursday, March 12th

Due date: Thursday, March 19th

1. Space garbage

In a convenient coordinate system, the spacetime of the earth is approximately

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 + \frac{2GM}{r}\right) \left[dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)\right] \\ &= -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 + \frac{2GM}{r}\right) (dx^2 + dy^2 + dz^2), \end{aligned}$$

where M is the earth's mass. In the second version, we've remapped the spherical coordinates to Cartesian coordinates in the usual way:

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta.$$

Note that the Cartesian form of the spacetime metric is conveniently written $g_{\alpha\beta} = \eta_{\alpha\beta} - 2\Phi \text{diag}(1, 1, 1, 1)$, with $\Phi \equiv -GM/r$. You may assume $|\Phi| \ll 1$ throughout this problem.

The space shuttle orbits the earth in a circular ($u^r = 0$), equatorial ($\theta = \pi/2, u^\theta = 0$) orbit of radius R .

(a) [5 pts] Using the geodesic equation, show that an orbit which begins equatorial remains equatorial: $du^\theta/dt = 0$ if $u^\theta = 0$ and $\theta = \pi/2$ at $t = 0$.

Hint 1: Begin by computing the non-zero connection coefficients. You can use the fact that the metric is diagonal and that $\Phi \ll 1$ to simplify your answer. The results of Carroll 3.3 should help; or, you can “brute force” it using the mathematica tools.

Hint 2 (which applies more directly to later parts of this problem): In the past, people have struggled a bit to take advantage of $|\Phi| \ll 1$. If you don't use this well, you'll find that the answers for the connection and the curvature tensors are pretty messy. Suggestion: If you do the calculation with mathematica, insert an *order counting parameter* ϵ whose formal value is 1. In other words, put

$$\begin{aligned} ds^2 &= -\left(1 - \epsilon \frac{2GM}{r}\right) dt^2 + \left(1 + \epsilon \frac{2GM}{r}\right) \left[dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)\right] \\ &= -\left(1 - \epsilon \frac{2GM}{r}\right) dt^2 + \left(1 + \epsilon \frac{2GM}{r}\right) (dx^2 + dy^2 + dz^2). \end{aligned}$$

Then, after computing connection coefficients and/or curvature, expand in ϵ and discard terms at $\mathcal{O}(\epsilon^2)$ or higher; the mathematica commands **Series** and **Normal** may be helpful here. (Comment: As we'll see after spring break, this line element is strictly correct only in a linearized approximation to general relativity. Hence, it is formally *wrong* to include terms beyond leading order in Φ anyway.)

We now require that the orbit must remain circular: $du^r/d\tau = 0$.

(b) [5 pts] By enforcing this condition with the geodesic equation, derive an expression for the orbital frequency

$$\Omega \equiv \frac{d\phi/d\tau}{dt/d\tau}.$$

Does the result look familiar?

The next part is most conveniently described in Cartesian coordinates; you may describe the shuttle's orbit as $x = R \cos \Omega t$, $y = R \sin \Omega t$.

An astronaut releases a bag of garbage into space, spatially displaced from the shuttle by $\xi^i = x_{\text{garbage}}^i - x_{\text{shuttle}}^i$.

(c) [10 pts] Using the equation of geodesic deviation, work out differential equations for the evolution of ξ^x , ξ^y , and ξ^z as a function of time. You may neglect terms in $(GM/r)^2$, and you may treat all orbital velocities as non-relativistic. You will need the Cartesian connection coefficients for this.

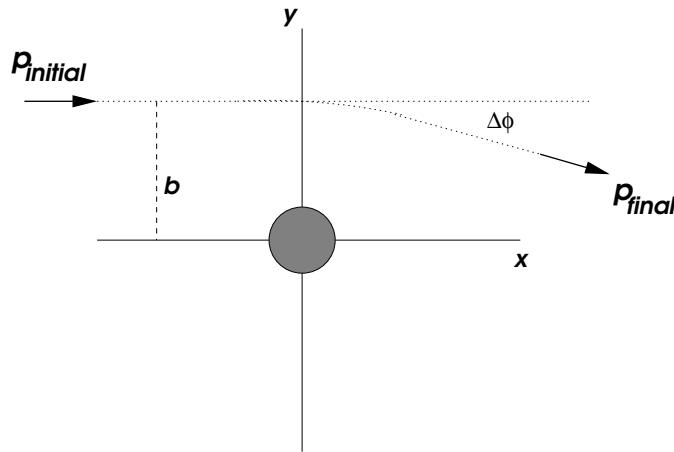
Hint 3: This is the point at which people tend to make this problem much harder than it needs to be. My suggestion: Expand this equation as much as you can without inserting explicit forms for the connection and for Riemann. Use the fact that the orbital velocity and the garbage's velocity is non-relativistic to simplify terms involving 4-velocities. (For example, the coupling to the Riemann tensor, $R^\mu{}_{\alpha\beta\nu}u^\alpha u^\beta \xi^\nu$ simplifies to $R^\mu{}_{00\nu}\xi^\nu$ in this limit.) Then use the smallness of Φ to simplify further. For example, working in Cartesian coordinates, you should find that connection squared terms are of order Φ^2 and hence die. You should find that the Riemann tensor simply looks like two partial derivatives of Φ — a particularly simple form.

Hint 4: Even after doing all the above simplification, you should find that the differential equation you work out couples spatial components of the displacement ξ^i to timelike components ξ^0 . I suggest treating these components separately: First, work out a differential equation that describes the evolution of ξ^0 . It should be of the form $d^2\xi^0/dt^2 = (\text{something which reduces to a really simple form})$. Then, impose the initial conditions $\xi^0 = 0$, $\partial_t \xi^0 = 0$ — in other words, the clocks of the shuttle and the garbage start synchronized (same initial time value, same initial tick rate). This will simplify things quite a bit.

(d) [5 pts] Suppose the initial displacement is $\xi^x = \xi^y = 0$, $\xi^z = L$, $d\xi^i/d\tau = 0$. (The equations you derived above will simplify even more in this case.) Has the astronaut succeeded in getting rid of the garbage?

2. [15 pts] Light bending

The spacetime of the sun can be written using the same line element as that of Problem 1 (substituting the sun’s mass for M). Consider a light ray which initially is moving purely along the x axis. Suppose that it passes near the sun with impact parameter (the distance between the axis passing through the sun’s center and the axis along which the light initially moves) b , as shown in this figure:



Using the geodesic equation, compute the angle by which the sun bends the light ray, $\Delta\phi = (p^y/p^x)_{\text{final}}$, expressing your answer using GM and b . You may assume that $GM/r \ll 1$ and that the bending angle is very small (so that $p_{\text{final}}^x \simeq p_{\text{initial}}^x$ to leading order in $\Delta\phi$).

Historical note: One can crudely calculate this effect in Newtonian theory by imagining that a photon has an effective mass $m_{\text{eff}} = \hbar\omega/c^2$ and asking for the transverse momentum deflection imparted by Newtonian gravity. The answer is exactly half the angle computed using the geodesic equation applied to the weak-field spacetime. The factor of two arises because of “spatial curvature” — the factor $(1 - 2\Phi)$ that multiplies the spatial sector of the spacetime metric. An eclipse observing expedition led by Arthur S. Eddington confirmed the relativistic prediction by carefully measuring stellar positions near the limb of the sun and comparing to their positions when the sun is in a different part of the sky. These results were instrumental in convincing the scientific world that general relativity describes gravity, and were largely responsible for Einstein becoming a household name.

It’s worth noting that the expedition’s data were not of particularly good quality; it’s something of a miracle that they were able to get such good agreement with the relativistic prediction. Indeed, some revisionists have claimed that Eddington fudged the data to get the answer that he “knew” was correct! See Daniel Kennefick’s article in the March 2009 issue of *Physics Today* for detailed discussion of this. Modern measurements using radar propagation in the solar system measure this bending angle with very high precision; the relativistic prediction has now been confirmed with a fractional precision $\sim 10^{-5}$, and not just around the sun — it’s measured with precision for many bodies in the solar system. High precision astrometry must take into account light bending due to the gravity of all bodies in the solar system to get accurate results.

Turning it around, one can *assume* the light bending result and use measurements of it to learn about a system's gravity. This is how gravitational lensing is done.

3. [10 pts] Parallel transport on a sphere

On the surface of a 2-sphere of radius a , $ds^2 = a^2 (d\theta^2 + \sin^2 \theta d\phi^2)$. Consider the vector $\vec{A}_0 = \vec{e}_\theta$ at $\theta = \theta_0$, $\phi = 0$. The vector is parallel transported all the way around the latitude circle $\theta = \theta_0$ (i.e, over the range $0 \leq \phi \leq 2\pi$ at $\theta = \theta_0$). What is the resulting vector \vec{A} ? What is its magnitude $(\vec{A} \cdot \vec{A})^{1/2}$? (Hint: derive differential equations for A^θ and A^ϕ as a function of ϕ .)

4. Curvature of a sphere

(a) [8 pts] Compute all the nonvanishing components of the Riemann tensor R_{ijkl} [$(i, j, k, l) \in (\theta, \phi)$] for the surface of a 2-sphere.

(b) [8 pts] Consider the parallel transport of a tangent vector $\vec{A} = A^\theta \vec{e}_\theta + A^\phi \vec{e}_\phi$ on the sphere around an infinitesimal parallelogram of sides $\vec{e}_\theta d\theta$ and $\vec{e}_\phi d\phi$. Using the results of part (a), show that to first order in $d\Omega \equiv \sin \theta d\theta d\phi$, the length of \vec{A} is unchanged, but its direction rotates through an angle equal to $d\Omega$.

(c) [8 pts] Show that, if \vec{A} is parallel transported around the boundary of any simply connected solid angle Ω on this sphere, its direction rotates through an angle Ω . (“Simply connected” is a topological term meaning that the boundary of the region could be shrunk to a point; it tells us that there are no holes in the manifold or other pathologies. See <http://mathworld.wolfram.com/SimpleConnected.html> for illustrations.) Using the result of part (b) and intuition from proofs of Stoke's Theorem, this should be an easy calculation. Compare with the result of Problem 3.

5. [10 pts] Killing vectors and curvature

Prove that the relations

$$\begin{aligned}\nabla_\nu \nabla_\mu \xi^\alpha &= R^\alpha{}_{\mu\nu\beta} \xi^\beta \\ \square \xi^\alpha &\equiv \nabla^\mu \nabla_\mu \xi^\alpha = -R^\alpha{}_\beta \xi^\beta\end{aligned}$$

are satisfied by any Killing vector ξ^α . This result is not too difficult to derive using the commutator $[\nabla_\alpha, \nabla_\beta] = \nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha$ and the identities

$$\begin{aligned}[\nabla_\alpha, \nabla_\beta] V^\mu &= R^\mu{}_{\nu\alpha\beta} V^\nu \\ [\nabla_\alpha, \nabla_\beta] V_\mu &= -R^\nu{}_{\mu\alpha\beta} V_\nu\end{aligned}$$

(If you are reading Schutz, these identities are incorrectly given on p. 171 — the minus sign is missing in the second identity. The “intuitive” explanation given for why there should be no minus sign is wrong.)

6. Riemann tensor for 1+1 static spacetimes

(a) [8 pts] Compute all the nonvanishing components of the Riemann tensor for the spacetime with line element

$$ds^2 = -e^{2\phi(x)} dt^2 + e^{-2\psi(x)} dx^2 .$$

(b) [8 pts] For the case $\phi = \psi = \frac{1}{2} \ln |g(x - x_0)|$ where g and x_0 are constants, show that the spacetime is flat and find a coordinate transformation to globally flat coordinates (\bar{t}, \bar{x}) such that $ds^2 = -d\bar{t}^2 + d\bar{x}^2$.