

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS
8.962 SPRING 2009

PROBLEM SET 7

Post date: Thursday, April 2nd

Due date: Thursday, April 9th

1. Gravitomagnetism

In lecture and working in Lorentz gauge, we examined the linearized Einstein field equations for a static source,

$$\square \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \quad \rightarrow \quad \nabla^2 \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} ,$$

where ∇^2 is the ordinary Euclidean 3-space Laplacian operator. For a static, non-relativistic source, the only non-zero stress-energy component is (to sufficient accuracy for our purposes)

$$T_{00} = \rho .$$

Using this, we found

$$\bar{h}_{00} = -4\Phi \rightarrow h_{\mu\nu} = -2\Phi \text{diag}(1, 1, 1, 1) ,$$

where $\Phi = -GM/r$ is the Newtonian gravitational potential.

We will now modify this slightly by imagining that the source rotates, and thus is characterized by a spin angular momentum with spatial components S^i as well as a mass M .

(a) [7 pts] Consider the source to be spherically symmetric, with uniform density ρ and radius R . Take it to be rotating rigidly about the $x^3 \equiv z$ axis with constant angular velocity Ω . Working in a Lorentz frame that is at rest with respect to the center of mass of the source, work out all components of the stress energy tensor $T_{\mu\nu}$ to first order in Ω . (Assume ρ , R , and Ω are constant.) Indicate which components would change if you included terms to second order in Ω , but don't calculate those second order corrections. (You may neglect pressure terms throughout your calculation.)

(b) [15 pts] Solve for the Cartesian off-diagonal components h_{0x} , h_{0y} , h_{0z} . (Note that $h_{0i} = \bar{h}_{0i}$ since trace reversal has no effect on off-diagonal components.)

This is a moderately challenging calculation. The following tips should help:

- Recall that the formal solution to the Poisson-type equation for h_{0i} is

$$h_{0i}(\mathbf{x}) = 4G \int \frac{T_{0i}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x'$$

where \mathbf{x} is the “field point”, the location of the point at which h_{0i} is to be evaluated, and \mathbf{x}' is the “source point”, a coordinate within the source over which the integral is taken. [Boldface quantities denote 3-vectors: $\mathbf{x} \doteq (x, y, z)$.]

- The following expansion for the factor $1/|\mathbf{x} - \mathbf{x}'|$ is very useful:

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{r} + \frac{x^j x^{j'}}{r^3} + \dots$$

You may assume this identity in your solution. Note also that a sum over j is implied here; we are allowed to be sloppy about the placement of indices since the spatial metric is δ_{ij} to leading order. [This identity is more often seen as an expansion in spherical harmonics; see, for example, J. D. Jackson, Sec. 3.6 (2nd edition). This form in terms of Cartesian coordinates is equivalent.]

- After you have set up your integral, convert the primed integration variable to spherical coordinates to do the integration:

$$\begin{aligned} x^{1'} &= x' \rightarrow r' \sin \theta' \cos \phi' \\ x^{2'} &= y' \rightarrow r' \sin \theta' \sin \phi' \\ x^{3'} &= z' \rightarrow r' \cos \theta' \end{aligned}$$

Your final metric components should be proportional to $\rho R^5/r^3$.

(c) [5 pts] Using the identity $S^i = I\Omega^i$ where I is moment of inertia and Ω^i is the i th component of the angular velocity vector, rewrite your answer in terms of the angular momentum S^i .

Although we derived this result for a special situation (uniform density, spherical body, rigid rotation), the result we obtain in terms of S^i is completely general; see, for example, MTW Sec. 19.1.

(d) [8 pts] Converting to spherical coordinates, find h_{0r} , $h_{0\theta}$, $h_{0\phi}$.

Hint: Only one of these components is non-zero. After changing coordinates, you should find that this non-zero component is $\propto S^z \sin^2 \theta/r$.

2. Comparison of linearized GR and Maxwell's theory

Consider the line element

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)(dx^2 + dy^2 + dz^2) + 2\beta_i dx^i dt ;$$

in other words, the usual weak field line element on the diagonal with $h_{0i} = \beta^i$. (To first order in deviations from flat spacetime, we also have $\beta^i = \delta^{ij}\beta_j = \beta_i$, and also $h^{0i} = -\beta^i$.)

(a) [10 pts] Show that the geodesic equation for a particle moving in this spacetime gives the following equation of motion to first order in the particle's velocity \mathbf{v} :

$$m \frac{d^2 \mathbf{x}}{dt^2} = m \mathbf{g} + m(\mathbf{v} \times \mathbf{H}) .$$

Here, \mathbf{x} is a 3-vector representing the position of the particle, and

$$\begin{aligned} \mathbf{g} &= -\nabla\Phi , \\ \mathbf{H} &= \nabla \times \boldsymbol{\beta} , \end{aligned}$$

where ∇ represents the ordinary gradient operator in Euclidean 3-space.

(b) [10 pts] Show that for stationary sources (i.e., no component of the stress energy tensor shows time variation) the Einstein field equations may be written

$$\begin{aligned}\nabla \cdot \mathbf{g} &= -4\pi G\rho, \\ \nabla \times \mathbf{H} &= -16\pi G\mathbf{J} \\ \nabla \cdot \mathbf{H} &= 0, \\ \nabla \times \mathbf{g} &= 0.\end{aligned}$$

The current $\mathbf{J} = \rho\mathbf{v}$, where \mathbf{v} is the velocity of fluid flow in the source. (Note that the second two equations follow from the definitions of \mathbf{g} and \mathbf{H} , so the only labor is in working out the first two.)

(c) [5 pts] These equations clearly bear a strong resemblance to Maxwell's equations in the limit $\partial_t\mathbf{E} = \partial_t\mathbf{B} = 0$; the main differences are the reversed sign in both equations, and the extra factor of 4 (compared to Maxwell) in the curl equation. Can you give a simple explanation for these differences?

3. [13 pts] Carroll: Chapter 7, Problem 1.

In this problem, Carroll asks us to vary a certain Lagrangian to construct the linearized Einstein tensor [Carroll Eq. (7.8)]. If you vary this Lagrangian in the "obvious" way, you will probably find that you get *almost* the correct Einstein tensor — you should get Eq. (7.8), but with the first two terms replaced with 2 times the first term. In other words, you don't get the symmetrization on μ and ν that we should get.

What is going on here? The issue is that the Lagrangian doesn't "know", *a priori*, that the tensor $h_{\mu\nu}$ is symmetric. This has a strong impact on the second term of the Lagrangian — it should be symmetric with respect to exchange of the indices ρ and σ , but isn't unless you somehow build in the knowledge we have of this symmetry.

There are two simple ways to address this:

- a. Rewrite the Lagrangian to force this symmetrization:

$$(\partial_\mu h^{\rho\sigma})(\partial_\rho h^\mu{}_\sigma) \rightarrow \frac{1}{2} [(\partial_\mu h^{\rho\sigma})(\partial_\rho h^\mu{}_\sigma) + (\partial_\mu h^{\rho\sigma})(\partial_\sigma h^\mu{}_\rho)];$$

- b. Make sure that, in our variation, this symmetry is enforced. The way I did this was to note that I should have

$$\begin{aligned}\frac{\delta(\partial_\mu h_{\rho\sigma})}{\delta(\partial_\gamma h_{\alpha\beta})} &= \frac{\delta(\partial_\mu h_{\rho\sigma})}{\delta(\partial_\gamma h_{\beta\alpha})} \\ &= \frac{1}{2} \left[\frac{\delta(\partial_\mu h_{\rho\sigma})}{\delta(\partial_\gamma h_{\alpha\beta})} + \frac{\delta(\partial_\mu h_{\rho\sigma})}{\delta(\partial_\gamma h_{\beta\alpha})} \right] \\ &= \frac{1}{2} \left[\delta^\gamma{}_\mu \delta^\alpha{}_\rho \delta^\beta{}_\sigma + \delta^\gamma{}_\mu \delta^\alpha{}_\sigma \delta^\beta{}_\rho \right].\end{aligned}$$

It shouldn't be too difficult to convince yourself that these methods are equivalent.

4. [12 pts] Carroll: Chapter 7, Problem 4.

Note: It's easy to get tripped up in this problem by thinking of x^μ as the components of a vector. Doing so will make the problem far more complicated than it needs to be. The idea instead is to regard the index μ as labeling members of a set of scalar functions: Each coordinate is just some scalar function, and our goal is to show that "box" applied to the scalar x^μ is equivalent to Lorentz gauge in linearized theory. In practice, this means you should ignore the fact that x^μ has an index when you expand the box operator. (As a notational mnemonic, you might want to replace x^μ with some scalar function f until you have expanded all the covariant derivatives into partials and connection coefficients.)