

MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
DEPARTMENT OF PHYSICS  
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PROBLEM SET 8

Post date: Thursday, April 9th

Due date: Thursday, April 16th

1. [15 pts] In lecture, we derived the following formula for the leading gravitational radiation generated by a source:

$$h_{ij}^{\text{TT}} = \frac{2}{r} \ddot{I}_{kl} \left( P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl} \right).$$

Here, overdot denotes  $d/dt$ ,  $I_{kl} = \int dV \rho x^k x^l$ , we are ignoring the distinction between upstairs and downstairs indices since we are in a nearly flat region and can work in nearly inertial coordinates, and the combination of projection tensors guarantees that the resulting tensor is transverse and traceless.

Show that the same result is obtained if one uses the traceless “quadrupole moment tensor”  $\mathcal{I}_{kl} = I_{kl} - \frac{1}{3} \delta_{kl} I$  instead of  $I_{jk}$ , where  $I = \delta_{kl} I_{kl}$ .

*Comment 1:* This is less trivial than it may seem since there are really two different trace operations defined here: A trace with respect to the spatial metric  $\eta_{ij} = \delta_{ij}$ , and a trace with respect to the metric of the subspace orthogonal to the propagation direction,  $P_{ij}$ .

*Comment 2:* A radiative  $l$ -pole has  $2l + 1$  separate components — a scalar has one component, a dipole has three, a quadrupole has five. The symmetric spatial tensor  $I_{ij}$  has 6 components, so there must be “extra” information there unrelated to radiation. This exercise proves that this extra information is bound up in the trace. For this reason, you will often see the formula for  $h_{ij}^{\text{TT}}$  written in terms of  $\mathcal{I}_{ij}$  rather than  $I_{ij}$ .

2. Binary system

Consider a binary consisting of two masses  $m_1$  and  $m_2$  in a circular orbit about one another; the separation of the bodies is  $R$ . Consider the orbit to be adequately described using Newtonian gravity; in this problem, we will use this description to compute the leading effects due to gravitational-wave emission. (We use Newtonian gravity to give the short-timescale orbital kinematics; we use the quadrupole gravitational waves to compute how this orbit evolves on long timescales.)

[*Hint:* Don’t forget that orbits in a problem of this type are most easily described using the “reduced system”: a body of mass  $\mu = m_1 m_2 / (m_1 + m_2)$  in circular orbit of radius  $R$  around a body of mass  $M = m_1 + m_2$ . If you need a refresher, this result is derived in all junior-level mechanics textbooks; see, for example, Goldstein Sec. 3.1.]

(a) [8 pts] Compute the gravitational-wave tensor  $h_{ij}^{\text{TT}}$  as measured by an observer looking down the angular momentum axis of the system (i.e., the  $z$ -axis if you define the orbital plane as the  $x - y$  plane).

(b) [7 pts] Compute the total rate at which energy is carried away from the system by gravitational waves,  $dE/dt$ .

Due to this loss of energy, the radius of the orbit will gradually shrink, and the frequency of the binary will “chirp” to higher frequencies as time passes.

(c) [7 pts] By asserting *global* conservation of energy in the following form,

$$\frac{d}{dt}(E_{\text{kinetic}} + E_{\text{potential}} + E_{\text{GW}}) = 0, \quad (1)$$

derive an equation for  $dr/dt$ , the rate at which the orbital radius shrinks.

[*Hint*: Don’t forget that for circular, Newtonian orbits, there is a simple relationship expressing  $E_{\text{kinetic}} + E_{\text{potential}}$  as a function of  $r$ , as well as a simple result for the orbital frequency as a function of  $r$  (Kepler’s 3rd law).]

(d) [6 pts] Derive the rate of change of the orbital angular frequency,  $\Omega$ , caused by gravitational-wave emission. You should find that the masses only appear in the combination  $\mathcal{M} \equiv \mu^{3/5} M^{2/5}$ , perhaps raised to some power. This combination of masses is known as the “chirp mass”, since it sets the rate at which the frequency “chirps”.

(e) [4 pts] Integrate the  $d\Omega/dt$  you get in part (d) to find  $\Omega(t)$ , the time evolution of the binary’s orbital frequency. Let  $T_{\text{coal}}$  (“coalescence time”) be the time at which the frequency (formally) goes to infinity. (In reality, various approximations we have introduced into this calculation break down before this;  $T_{\text{coal}}$  is nonetheless a fairly accurate stand-in for the time at which the members of a binary merge due to gravitational-wave emission.) Your answer should be a power law in  $T_{\text{coal}} - t$ .

*Comment*: Modulo a correction to account for the effect of eccentricity, the  $\Omega(t)$  you just derived is what is observed in the orbit of relativistic binary pulsars. Measuring this decay and showing that it agreed with general relativity’s prediction is in part what led to the Nobel Prize for Russell Hulse and Joe Taylor in 1993.

### 3. Wave equation for the Riemann tensor in linearized theory

As we have emphasized from time to time, there is a nice analogy between the metric of GR and the electromagnetic potential, and likewise between curvature tensors and the electromagnetic field. This suggests that it should be possible to build a wave equation for the curvature tensor, and that it might give a nicely gauge-invariant way to regard various things.

As background to this problem and the next one, recall that the Einstein field equations can be written in the trace-reversed form

$$R^\mu{}_{\alpha\mu\beta} = R_{\alpha\beta} = 8\pi\bar{T}_{\alpha\beta}$$

where  $\bar{T}_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T^\gamma{}_\gamma$ . (Proof: Just trace reverse  $G_{\alpha\beta} = 8\pi GT_{\alpha\beta}$ .)

To keep things simple, we begin with linearized theory, working in nearly inertial coordinates:  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$  with  $\|h_{\mu\nu}\| \ll 1$ .

(a) [5 pts] To linear order in  $h$ , the Bianchi identity can be written

$$\partial_\alpha R_{\beta\gamma\mu\nu} + \partial_\beta R_{\gamma\alpha\mu\nu} + \partial_\gamma R_{\alpha\beta\mu\nu} = 0.$$

Using this equation, show that the divergence of the Riemann tensor is related to the gradient of the trace-reversed stress energy tensor:

$$\partial_\alpha R^\alpha{}_{\beta\gamma\delta} = 8\pi G \left[ \text{source involving gradient of } \bar{T}_{\mu\nu} \right]$$

(b) [6 pts] Now use the Bianchi identity and the solution to part (a) to develop a wave equation for the Riemann tensor of the form

$$\square R_{\alpha\beta\mu\nu} = 8\pi G [\text{source involving double gradients of } \bar{T}_{\mu\nu}.]$$

Solve this equation (formally) with the radiative Green's function introduced in lecture.

(c) [7 pts] Now specialize to a plane gravitational wave propagating in the  $z$ -direction through vacuum. The corresponding solution to the above wave equation takes the form  $R_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu}(t-z)$ . Using the Bianchi identity and the symmetries of Riemann, show that the only non-zero independent components of the Riemann tensor are of the form  $R_{i0j0}$ ; all other components are either zero or are trivially related by symmetries. Note, this analysis will be greatly facilitated by expanding the Riemann tensor in plane waves: Put

$$R_{\alpha\beta\mu\nu} = C_{\alpha\beta\mu\nu} e^{ik_\sigma x^\sigma}$$

with

$$k^\sigma \doteq (\omega, 0, 0, \omega).$$

This changes the Bianchi identity into a simple algebraic relation between components of Riemann and components  $k^\sigma$ .

(d) [4 pts] Show that the only independent  $R_{i0j0}$  are  $R_{x0x0}(t-z) = -R_{y0y0}(t-z)$  and  $R_{x0y0}(t-z) = R_{y0x0}(t-z)$ ; any other non-zero components are trivially related by Riemann symmetries. (Note, since  $\partial_t = -\partial_z$  for this spacetime, components involving the  $z$  components like  $R_{xzxz}$  can also be regarded as “trivially” related.) The first non-zero components correspond to the  $+$  polarization discussed in lecture; the second corresponds to the  $\times$  polarization.

(e) [5 pts] Define fields  $h_+(t-z)$  and  $h_\times(t-z)$  in terms of these components of the Riemann tensor by

$$R_{x0x0} = -\frac{1}{2}\partial_t^2 h_+, \quad R_{x0y0} = -\frac{1}{2}\partial_t^2 h_\times.$$

Also recall the expression for the Riemann tensor in linearized theory:

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}(\partial_\alpha\partial_\nu h_{\beta\mu} + \partial_\beta\partial_\mu h_{\alpha\nu} - \partial_\alpha\partial_\mu h_{\beta\nu} - \partial_\beta\partial_\nu h_{\alpha\mu})$$

Comparing these two forms, show that  $h_+ = h_{xx}^{\text{TT}} = -h_{yy}^{\text{TT}}$ ,  $h_\times = h_{xy}^{\text{TT}} = h_{yx}^{\text{TT}}$ .

(f) [6 pts] Show that when one rotates the coordinate system about the waves' propagation direction (i.e., about the  $z$ -axis) by an angle  $\theta$  [so that  $x' + iy' = (x + iy)e^{-i\theta}$ ], the gravitational-wave fields  $h_+$  and  $h_\times$  transform such that

$$h'_+ + ih'_\times = (h_+ + ih_\times)e^{-i2\theta}$$

This equation is the classical foundation of the statement that the graviton is spin 2: it tells us that the quantity  $h_+ + ih_\times$  has “spin-weight” 2.

4. Nonlinear wave equation for the Riemann tensor

Begin now with the full Bianchi identity:

$$\nabla_\alpha R_{\beta\gamma\mu\nu} + \nabla_\beta R_{\gamma\alpha\mu\nu} + \nabla_\gamma R_{\alpha\beta\mu\nu} = 0 .$$

(a) [10 pts] Develop the covariant analog to your answer to part (a) of problem (3):

$$\nabla_\alpha R^\alpha{}_{\beta\gamma\delta} = 8\pi G \left[ \text{source involving covariant gradient of } \bar{T}_{\mu\nu} . \right]$$

(b) [10 pts] Using the Bianchi identity and the solution to part (a), develop a nonlinear wave equation for the Riemann tensor of the form

$$\square R^\alpha{}_{\beta\gamma\delta} = 8\pi G [\text{source}] .$$

In this case, the source should involve three kinds of terms: double covariant gradients of  $\bar{T}_{\mu\nu}$ , coupling of  $\bar{T}_{\mu\nu}$  to the Riemann tensor, and coupling of Riemann to Riemann. Here, the operator  $\square = \nabla^\mu \nabla_\mu$ .

*Hint:* A key element of your solution will be the fact that the commutator of two covariant derivatives generates a Riemann curvature tensor. In particular, the following identity will be useful:

$$[\nabla_\mu, \nabla_\nu] R^\alpha{}_{\beta\gamma\rho} = R^\alpha{}_{\sigma\mu\nu} R^\sigma{}_{\beta\gamma\rho} - R^\sigma{}_{\beta\mu\nu} R^\alpha{}_{\sigma\gamma\rho} - R^\sigma{}_{\gamma\mu\nu} R^\alpha{}_{\beta\sigma\rho} - R^\sigma{}_{\rho\mu\nu} R^\alpha{}_{\beta\gamma\sigma} .$$

This isn't as bad as it looks! The trick will be to use this to manipulate the result you get as you develop  $\square R^\alpha{}_{\beta\gamma\delta}$  so that you get divergences of Riemann [for which the result of part (a) applies] and Riemann-squared-type terms.

*Comment:* The structure of this equation is *much* more complicated. "Normal" source coupling to curvature and curvature coupling to curvature both act as sources in this equation. This is an explicit manifestation of general relativity's nonlinearity. The non-linear wave equation you develop here was first found by Roger Penrose, and is sometimes called the Penrose wave equation. A variant based on an expansion around the Riemann tensor for a black hole spacetime was developed by Saul Teukolsky and has played an extremely important role in astrophysical relativity research.