8.962 PS 1 Solutions

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1. (a) [5 pts] Show that the sum of any two orthogonal (scalar product is zero) spacelike vectors is spacelike.

   Solution: If we have two spacelike vectors, $\vec{u}$ and $\vec{v}$, which are orthogonal, then we know the following about the various inner products we can construct:

$$\vec{u} \cdot \vec{u} \equiv u^2 > 0 \quad (1)$$

$$\vec{v} \cdot \vec{v} \equiv v^2 > 0 \quad (2)$$

$$\vec{u} \cdot \vec{v} = 0. \quad (3)$$

   Let $\vec{w} = \vec{u} + \vec{v}$. Then

$$\vec{w} \cdot \vec{w} = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = u^2 + 2\vec{u} \cdot \vec{v} + v^2 > 0, \quad (4)$$

   so $\vec{w}$ is spacelike, too.

   (b) [5 pts] Show that a timelike vector and a null vector cannot be orthogonal.

   Solution: If we have a timelike vector, $\vec{u}$, we know that we can find a coordinate system in which $\vec{u} \doteq \left(u^0, 0, 0, 0\right)$. Further, we can choose this coordinate system such that the null vector $\vec{v} \doteq \left(v^0, v^0, 0, 0\right)$. Note that both $u^0 \neq 0$ and $v^0 \neq 0$. Then

$$\vec{u} \cdot \vec{v} = u^\mu v_\mu = -u^0 v^0 \neq 0. \quad (5)$$

   Because $\vec{u} \cdot \vec{v} \neq 0$, $\vec{u}$ and $\vec{v}$ are not orthogonal.

2. In some reference frame, the vector fields $\vec{U}$ and $\vec{D}$ have the components

$$U^\alpha \doteq (1 + 2t^2, 2t^2, -2t, 0) \quad (6)$$

$$D^\alpha \doteq (2x, 3tx, \sqrt{5}t, 0) \,. \quad (7)$$

   where t, x, and y are the usual Cartesian coordinates in the specified reference frame. The scalar ρ has the value

$$\rho = t^2 + x^2 - y^2 \,. \quad (8)$$

(The relationship "LHS $\doteq$ RHS" means "the object on the left-hand side is represented by the object on the right-hand side in the specified reference frame.")

(a) [3 pts] Show that $\vec{U}$ is suitable as a 4-velocity. Is $\vec{D}$?

Solution: To be suitable as a 4-velocity, a vector must have magnitude -1. We see that $\vec{U}$ does:

$$U^2 = \eta_{\mu\nu} U^\mu U^\nu = -\left(1+2t^2\right)^2 + 4t^4 + 4t^2 = -1. \tag{9}$$

Unfortunately, $\vec{D}$ does not:

$$D^2 = \eta_{\mu\nu} D^\mu D^\nu = -4x^2 + 9t^2x^2 + 5t^2, \tag{10}$$

which is not identically -1. (If we happen to restrict ourselves to the three-dimensional sub-manifold of spacetime where $-4x^2 + 9t^2x^2 + 5t^2 = -1$, then $\vec{D}$ is a suitable four-velocity. However, it is not a four-velocity for all of spacetime.)

(b) [3 pts] Find the spatial velocity $\mathbf{v}$ of a particle whose 4-velocity is $\vec{U}$, for arbitrary t. Describe the motion in the limits $t = 0$ and $t \to \infty$.

Solution: A four-velocity can be written as $(\gamma, \gamma\mathbf{v})$, so we have

$$v^i = \frac{U^i}{U^0} = \left(\frac{2t^2}{1+2t^2}, \frac{-2t}{1+2t^2}, 0\right). \tag{11}$$

At $t = 0$, we have $\mathbf{v} = \mathbf{0}$. The velocity will initially increase most rapidly in the y direction, and then, as $t \to \infty$, will asymptote to $\mathbf{v} = \hat{\mathbf{x}}$.

(c) [3 pts] Find $\partial_\beta U^\alpha$ for all α, β. Show that $U_\alpha \partial_\beta U^\alpha = 0$. (There's a clever way to do this, which you are welcome to point out. Please do it the brute force way as well as practice manipulating quantities like this.)

Solution: Representing $\partial_\beta U^\alpha$ as a matrix, we have

$$\partial_\beta U^\alpha \doteq \begin{pmatrix} 4t & 4t & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{12}$$

(Recall that β indexes rows, and α indexes columns.) Because the components of $\vec{U}$ depend only on t, only terms of the form $\partial_0 U^\alpha$ are nonzero. We find that

$$U_\alpha \partial_\beta U^\alpha \doteq \left(-4t\left(1+2t^2\right) + 8t^3 + 4t, 0, 0, 0\right) = (0, 0, 0, 0). \tag{13}$$

(Note that only the first component required any computation.)

A labor-saving realization is that

$$U_\alpha \partial_\beta U^\alpha = (1/2)\partial_\beta \left(U_\alpha U^\alpha\right) \equiv 0 \tag{14}$$

because the norm of $\vec{U}$ is independent of t, x, y, and z.

(d) [3 pts] Find $\partial_\alpha D^\alpha$.

Solution: $\partial_\alpha D^\alpha = 3t$.

(e) [3 pts] Find $\partial_\beta(U^\alpha D^\beta)$ for all α.

Solution: We have

$$\partial_\beta \left(U^\alpha D^\beta\right) = U^\alpha \partial_\beta D^\beta + D^\beta \partial_\beta U^\alpha. \tag{15}$$

We can compute this sum by reference to previous results from parts (c) and (d). The result is

$$\begin{aligned}\partial_\beta \left(U^\alpha D^\beta\right) &\doteq 3t\left(1+2t^2, 2t^2, -2t, 0\right) \\ &\quad + (2x, 3tx, \sqrt{5}t, 0)^T \cdot \begin{pmatrix} 4t & 4t & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \left(6t^3 + 8tx + 3t, 6t^3 + 8tx, -6t^2 - 4x, 0\right). \end{aligned} \tag{16}$$

(f) [3 pts] Find $U_\alpha \partial_\beta(U^\alpha D^\beta)$. Why is the answer so similar to that for (d)?

Solution: Contracting $\vec{U}$ with the result from part (e) gives

$$\begin{aligned} U_\alpha \partial_\beta(U^\alpha D^\beta) &= -\left(1+2t^2\right)\left(6t^3 + 8tx + 3t\right) \\ &\quad + 2t^2\left(6t^3 + 8tx\right) - 2t\left(-6t^2 - 4x\right) = -3t. \end{aligned} \tag{17}$$

This must be so because

$$U_\alpha \partial_\beta(U^\alpha D^\beta) = \partial_\beta\left(U_\alpha U^\alpha D^\beta\right) - U^\alpha D^\beta \partial_\beta U_\alpha. \tag{18}$$

The second term is zero (see (c)), and the first reduces to

$$U_\alpha \partial_\beta(U^\alpha D^\beta) = -\partial_\beta D^\beta, \tag{19}$$

which has been computed in (d).

(g) [3 pts] Calculate $\partial_\alpha \rho$ for all α. Calculate $\partial^\alpha \rho$.

Solution: We have

$$\partial_\mu \rho \doteq (2t, 2x, -2y, 0)\,. \tag{20}$$

Raising the index (contracting with η), we obtain

$$\partial^\mu \rho \doteq (-2t, 2x, -2y, 0)\,. \tag{21}$$

(h) [3 pts] Find $\nabla_{\vec{U}}\rho$ and $\nabla_{\vec{D}}\rho$.

Solution: We have

$$\nabla_{\vec{U}}\rho = U^\mu \partial_\mu \rho = 2t + 4t^3 + 4t^2 x + 4yt \tag{22}$$

and

$$\nabla_{\vec{D}}\rho = D^\mu \partial_\mu \rho = 4tx + 6x^2 t - 2\sqrt{5}yt \tag{23}$$

3. Consider a timelike unit 4-vector $\vec{U}$ and the tensor

$$P_{\alpha\beta} = \eta_{\alpha\beta} + U_\alpha U_\beta\ . \tag{24}$$

Show that this tensor is a projection operator that projects an arbitrary vector $\vec{V}$ into one orthogonal to $\vec{U}$. In other words, show that the vector $\vec{V}_\perp$ whose components are

$$V^\alpha_\perp = P^\alpha{}_\beta V^\beta \tag{25}$$

is

(a) [5 pts] orthogonal to $\vec{U}$

Solution:

$$\begin{aligned} U_\alpha V^\alpha_\perp &= U_\alpha P^\alpha{}_\beta V^\beta = U^\alpha P_{\alpha\beta} V^\beta \\ &= \eta_{\alpha\beta} U^\alpha V^\beta + U^\alpha U_\alpha U_\beta V^\beta = U^\alpha V_\alpha - U^\beta V_\beta = 0 \end{aligned} \tag{26}$$

(b) [5 pts] unaffected by further projections:

$$V^\alpha_{\perp\perp} \equiv P^\alpha{}_\beta V^\beta_\perp = V^\alpha_\perp\ . \tag{27}$$

Solution: Note that

$$P^\alpha{}_\beta = \eta^{\alpha\gamma} P_{\gamma\beta} = \delta^\alpha{}_\beta + U^\alpha U_\beta. \tag{28}$$

Applying this, we have

$$
\begin{aligned}
P^\alpha{}_\beta V_\perp^\beta &= P^\alpha{}_\beta P^\beta{}_\gamma V^\gamma \\
&= \left(\delta^\alpha{}_\beta \delta^\beta{}_\gamma + \delta^\alpha{}_\beta U^\beta U_\gamma + U^\alpha U_\beta \delta^\beta{}_\gamma + U^\alpha U_\beta U^\beta U_\gamma\right) V^\gamma \\
&= \left(\delta^\alpha{}_\gamma + U^\alpha U_\gamma\right) V^\gamma = P^\alpha{}_\gamma V^\gamma = V_\perp^\alpha. \qquad (29)
\end{aligned}
$$

(c) [5 pts] Show that $P_{\alpha\beta}$ is the metric for the space of vectors orthogonal to $\vec{U}$:

$$
P_{\alpha\beta} V_\perp^\alpha W_\perp^\beta = \vec{V}_\perp \cdot \vec{W}_\perp \ . \qquad (30)
$$

Solution: Consider

$$
P_{\alpha\beta} V_\perp^\alpha W_\perp^\beta = V_\perp^\alpha \eta_{\alpha\gamma} P^\gamma{}_\beta W_\perp^\beta. \qquad (31)
$$

But, $\vec{W}_\perp$ is unaffected by the projection (see (b)), so we have

$$
P_{\alpha\beta} V_\perp^\alpha W_\perp^\beta = V_\perp^\alpha \eta_{\alpha\gamma} W_\perp^\gamma = \vec{V}_\perp \cdot \vec{W}_\perp. \qquad (32)
$$

(d) [5 pts] Show that for an arbitrary nonnull vector $\vec{q}$, the projection tensor is given by

$$
P_{\alpha\beta}(q^\alpha) = \eta_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^\gamma q_\gamma} \ . \qquad (33)
$$

Do we need a projection tensor for null vectors?

Solution: We must verify the properties in (a) and (b) for the general $P^\alpha{}_\beta(\vec{q})$; then the property in (c) will follow automatically (we never used that $U^\alpha U_\alpha = -1$ in the proof of (c)). First we show that $q^\alpha P_{\alpha\beta}(\vec{q}) = 0$:

$$
q^\alpha P_{\alpha\beta}(\vec{q}) = q_\beta - \frac{q^\alpha q_\alpha q_\beta}{q^\gamma q_\gamma} = q_\beta - q_\beta \equiv 0. \qquad (34)
$$

Then, we show that $P^\alpha{}_\beta(\vec{q}) P^\beta{}_\gamma(\vec{q}) = P^\alpha{}_\gamma(\vec{q})$:

$$
\begin{aligned}
P^\alpha{}_\beta(\vec{q}) P^\beta{}_\gamma(\vec{q}) &= \left(\delta^\alpha{}_\beta - \frac{q^\alpha q_\beta}{q^\rho q_\rho}\right)\left(\delta^\beta{}_\gamma - \frac{q^\beta q_\gamma}{q^\sigma q_\sigma}\right) \\
&= \delta^\alpha{}_\beta \delta^\beta{}_\gamma - \delta^\alpha{}_\beta \frac{q^\beta q_\gamma}{q^\sigma q_\sigma} - \delta^\beta{}_\gamma \frac{q^\alpha q_\beta}{q^\rho q_\rho} + \frac{q^\alpha q_\beta q^\beta q_\gamma}{\left(q^\rho q_\rho\right)^2} \\
&= \delta^\alpha{}_\gamma - \frac{q^\alpha q_\gamma}{q^\rho q_\rho} = P^\alpha{}_\gamma(\vec{q}). \qquad (35)
\end{aligned}
$$

The above procedure will not work for null vectors (because the denominator in the second term of $P^{\alpha}{}_{\beta}$ becomes zero).

To (partially) address the question of whether we "need" a projector for null vectors, consider an arbitrary null vector $\vec{v}$. A potential projection tensor, $P_{\alpha\beta}(\vec{v})$, must have the appropriate transformation properties under Lorentz transformations. This means we must construct it using a linear combination of the available Lorentz tensors with the appropriate symmetry properties ($\eta_{\alpha\beta}$ and $v_\alpha v_\beta$) with Lorentz scalar coefficients:

$$P_{\alpha\beta}(\vec{v}) = f(v)\eta_{\alpha\beta} + g(v)v_\alpha v_\beta, \tag{36}$$

with f and g scalar functions yet to be determined. Requiring that $v^\alpha P_{\alpha\beta} = 0$ implies that $f(v) \equiv 0$. But then

$$P^{\alpha}{}_{\gamma}P^{\gamma}{}_{\beta} = P^{\alpha}{}_{\gamma}\eta^{\gamma\delta}P_{\delta\beta} = g^2(v)v^\alpha v_\gamma \eta^{\gamma\delta} v_\delta v_\beta = 0, \tag{37}$$

so $P^{\alpha}{}_{\gamma}P^{\gamma}{}_{\beta} = P^{\alpha}{}_{\beta}$ implies that $g(v) \equiv 0$. No non-zero, Lorentz-covariant projection operator onto the orthogonal subspace of a null vector exists. Hopefully, we don't need one!

Also, note that a null vector is orthogonal to itself. This means that the orthogonal subspace to a null vector *contains that vector*. Something weird is going on here with null vectors!

(This question could have other possible answers.)

4. [15 pts] Let $\Lambda_B(\mathbf{v})$ be a Lorentz boost associated with 3-velocity $\mathbf{v}$. Consider

$$\Lambda \equiv \Lambda_B(\mathbf{v}_1)\cdot\Lambda_B(\mathbf{v}_2)\cdot\Lambda_B(-\mathbf{v}_1)\cdot\Lambda_B(-\mathbf{v}_2) \tag{38}$$

where $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. Assume $v_1 \ll 1$, $v_2 \ll 1$.

Show that Λ is a rotation. What is the axis of rotation? What is the angle of rotation?

Solution: Without loss of generality, we can choose a coordinate system which has $\mathbf{v}_1 \doteq (v_1, 0, 0)$ and $\mathbf{v}_2 \doteq (0, v_2, 0)$—given any coordinate system, a rotation and possibly a parity inversion will produce the coordinate system required. Then we have

$$\Lambda_B(\pm\mathbf{v}_1) = \begin{pmatrix} 1+\frac{v_1^2}{2} & \mp v_1 & 0 & 0 \\ \mp v_1 & 1+\frac{v_1^2}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \mathcal{O}\left(v_1^3\right) \tag{39}$$

and

$$\Lambda_B(\pm\mathbf{v}_2) = \begin{pmatrix} 1+\frac{v_2^2}{2} & 0 & \mp v_2 & 0 \\ 0 & 1 & 0 & 0 \\ \mp v_2 & 0 & 1+\frac{v_2^2}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \mathcal{O}\left(v_2^3\right). \tag{40}$$

Multiplying out the matrices, we see that

$$\Lambda_B(\mathbf{v}_1)\cdot\Lambda_B(\mathbf{v}_2)\cdot\Lambda_B(-\mathbf{v}_1)\cdot\Lambda_B(-\mathbf{v}_2) \doteq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & v_1 v_2 & 0 \\ 0 & -v_1 v_2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{41}$$

to second-order in v_1 and v_2. We recognize this as a rotation described by the vector $\mathbf{v_2}\times\mathbf{v_1} \doteq -v_1 v_2 \hat{\mathbf{z}}$ in our special coordinate system. (The rotation described by a vector is a right-handed rotation about the axis defined by that vector with angle equal to the vector's magnitude.) Because our special coordinate system is related to any arbitrary coordinate system by a rotation and possibly a parity inversion, and $\mathbf{v_2}\times\mathbf{v_1}$ is covariant with respect to these operations, we see that

$$\Lambda_B(\mathbf{v}_1)\cdot\Lambda_B(\mathbf{v}_2)\cdot\Lambda_B(-\mathbf{v}_1)\cdot\Lambda_B(-\mathbf{v}_2) = R(\mathbf{v}_2\times\mathbf{v}_1) \tag{42}$$

5. "Superluminal" motion

The quasar 3C 273 emits relativistic blobs of plasma from near the massive black hole at its center. The blobs travel at speed v along a jet making an angle θ with respect to the line of sight of the observer. Projected onto the sky, the blobs appear to travel perpendicular to the line of sight with angular speed $v_{\rm app}/r$ where r is the distance to 3C 273 as and $v_{\rm app}$ is the apparent speed.

(a) [7 pts] Show that

$$v_{\rm app} = \frac{v\sin\theta}{1 - v\cos\theta}\ . \tag{43}$$

Solution: Here is the sequence of events: at $(t,x,y) = (0,0,0)$ a blob is emitted from the quasar. At $(t_0, vt_0\cos\theta, vt_0\sin\theta)$ the quasar emits a photon. At $(t, r, 0)$ the telescope on earth receives the photon. The observer projects the motion of the blob onto the y-axis, and believes that the blob has traveled a distance $L_p \equiv vt_0\sin\theta$. This geometry is illustrated in Figure 1. We need to re-express this in terms of the reception time of the photons, t, because the observer will observe an apparent velocity

$$v_{\rm app} = \frac{dL_p}{dt}. \tag{44}$$

We can relate t and t_0 using the light travel time to the observer:

$$t - t_0 = \sqrt{(r - vt_0\cos\theta)^2 + v^2t_0^2\sin^2\theta} \approx r - vt_0\cos\theta, \tag{45}$$

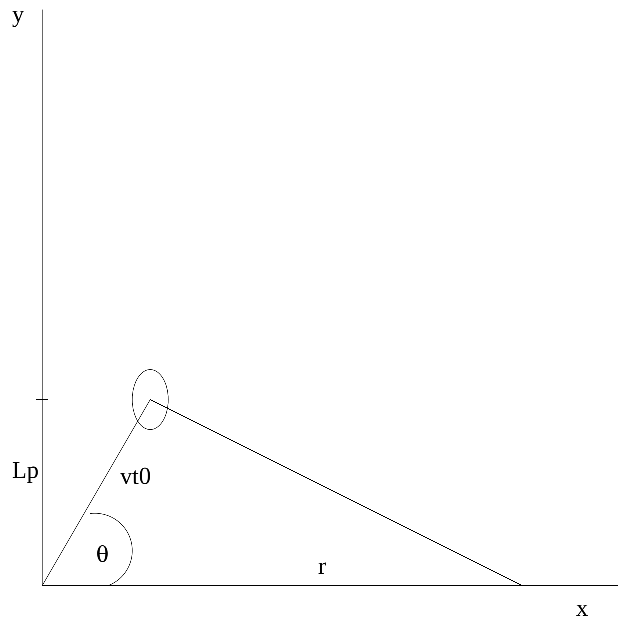

Figure 1: The quasar geometry in problem 5.

when $vt_0 \ll r$. Using this relation, we find

$$v_{\rm app} = \frac{dL_p}{dt} = \frac{v \sin\theta}{1 - v\cos\theta} \tag{46}$$

Note that this derivation is completely "Newtonian" in character (assuming only that light travels at a finite speed).

(b) [5 pts] For a given value of v, what value of θ maximizes $v_{\rm app}$? What is the corresponding maximal value of $v_{\rm app}$? Can this be greater than the speed of light? If so, is special relativity violated?

Solution: The extremizing condition is

$$\frac{dv_{\rm app}}{d\theta} = 0, \tag{47}$$

which occurs for $\theta = \cos^{-1} v$. The extreme value of $v_{\rm app}$ is

$$v_{\rm app}^{\rm max} = \frac{v}{\sqrt{1 - v^2}}. \tag{48}$$

$v_{\rm app}^{\rm max}$ can be arbitrarily large; this does not violate special relativity because $v_{\rm app}^{\rm max}$ is not the true velocity of any physical object.

(c) [3 pts] For 3C 273, $v_{\rm app} \simeq 10c$. What is the largest possible value of θ (in degrees)?

Solution: We have the relation

$$v = \frac{10}{10\cos\theta + \sin\theta}. \tag{49}$$

This implies that $d\theta/dv = (dv/d\theta)^{-1} \neq 0$ over the entire range of $\theta \in [0, \pi/2]$ and $v \in [0, 1]$, so the maximum value of θ will be attained at the boundary (i.e. when $v = 1$). When $v = 1$, we have

$$\theta_{\rm max} = \cos^{-1}\frac{99}{101}. \tag{50}$$

6. GZK cutoff in the cosmic ray spectrum

(a) [8 pts] Calculate the threshold energy of a nucleon N for it to undergo the reaction $\gamma + N \to N + \pi^0$, where γ represents a microwave background photon of energy kT with $T = 2.73$ K. Assume the collision is head-on and take the nucleon and pion masses to be 938 MeV and 135 MeV, respectively.

Solution: We know that these collisions conserve four-momentum, so we have

$$\vec{p}_{N_1} + \vec{p}_\gamma = \vec{p}_{N_2} + \vec{p}_{\pi^0}. \tag{51}$$

We will exploit that the squared magnitude of a four-vector is a Lorentz scalar to compute the magnitude of each side of this equation in a different frame. We compute the magnitude of the RHS in the center-of-momentum frame where the reaction products emerge at rest. In this frame,

$$\vec{p}_{N_2} \doteq (m_N, \mathbf{0}) \tag{52}$$

$$\vec{p}_{\pi^0} \doteq (m_{\pi^0}, \mathbf{0}), \tag{53}$$

and we have

$$(p_{N_2} + p_{\pi^0})^2 = -(m_N + m_{\pi^0})^2. \tag{54}$$

We will evaluate the LHS of equation (51) in the rest-frame of the CMB, where $E_\gamma = |\mathbf{p}_\gamma| = 2.73$ K. We rotate coordinate axes so that the three-momenta of the two particles lie on one of the spatial axes. We have

$$(p_{N_1} + p_\gamma)^2 = -m_N^2 - 2E_\gamma E_{N_1} + 2E_\gamma\sqrt{E_{N_1}^2 - m_N^2}. \tag{55}$$

Equating the magnitudes of the two sides of equation (51) and solving for E_{N_1} gives

$$E_{N_1} = \frac{4E_\gamma^2 m_N^2 + m_{\pi^0}^2 (2m_N + m_{\pi^0})^2}{4E_\gamma m_{\pi^0} (2m_N + m_{\pi^0})} \approx 2.9 \times 10^{11}\,\mathrm{GeV} \tag{56}$$

(b) [5 pts] Explain why one might expect to observe very few cosmic rays of energy above $\sim 10^{11}$ GeV.

Solution: If cosmic rays are truly cosmic, then, above 10^{11} GeV, they would react with CMB photons to produce pions, shedding energy until they cross the cutoff from part (a).

It turns out, when all relevant cross-sections (not just $p\gamma \to p\pi$) are considered, to be very hard to produce a model where significant numbers of $E > 10^{11}$ GeV cosmic rays travel more than a few tens of Mpc [1]. For reference, the nearest galaxy (Andromeda) is about 1 Mpc from the milky way.

(c) [3 pts] This expectation is called the Griesen-Zatsepin-Kuzmin (GZK) cutoff. Modern observations are consistent with this cutoff, but do find that there are a handful of rare cosmic rays that exceed this energy (sometimes significantly). Can you suggest a mechanism by which the GZK cutoff can be avoided?

Solution: If the high energy cosmic rays are produced locally (closer than a few tens of Mpc), then the GZK cutoff would not apply.

References

[1] A. Olinto, "Ultra high-energy cosmic rays: The theoretical challenge," Physics Reports **333**, 329 (2000).