8.962 Problem Set 3 Solutions

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1. Accelerated observer revisited:

   (a) [5 pts] Calculate the acceleration experienced by a CSO. You may find it helpful to use the fact that, along a CSO's trajectory in the accelerated coordinate system, $ds^2 = -d\tau^2$, where $d\tau$ is proper time along that trajectory.

   Solution: Let $\xi(\tau)$ be the trajectory of the CSO associated with $\bar{x}$. In the barred coordinate system, we have

   $$\xi^{\bar{\mu}} \doteq (A(\tau), \bar{x}). \tag{1}$$

   (This is the definition of a CSO—such an observer remains at fixed spatial coordinate.) But, we know that

   $$g_{\bar{\mu}\bar{\nu}}\dot{\xi}^{\bar{\mu}}\dot{\xi}^{\bar{\nu}} = -1 = -(g\bar{x})^2 \left(\dot{A}(\tau)\right)^2, \tag{2}$$

   where overdot denotes $d/d\tau$, and where we have used the components of the metric in the barred coordinates which we derived last week. This implies that

   $$A(\tau) = \frac{\tau}{g\bar{x}}, \tag{3}$$

   since $A(0) = \bar{t}(\tau = 0) = 0$.

   It seems tempting to try to compute

   $$a^{\bar{\mu}} = \frac{d^2}{d\tau^2}\xi^{\bar{\mu}}, \tag{4}$$

   but *this is not correct*! $\dot{\xi}^{\bar{\mu}}$ is a four-vector, and therefore, differentiating it properly requires using a covariant derivative. Physically, this is because as we move along the accelerated worldline, the basis objects are changing as well as the vector's components.

   Rather than compute the connection coefficients for the barred coordinate system, we will instead transform $\dot{\xi}$ into the un-barred (Lorentz) frame, where the connection vanishes. We have

$$\begin{aligned}
\dot{\xi}^{\mu}(\tau) &= \left.\frac{\partial x^{\mu}}{\partial x^{\bar{\mu}}}\right|_{\xi(\tau)} \dot{\xi}^{\bar{\mu}}(\tau) \\
&= \frac{1}{g\bar{x}}\left(g\bar{x}\cosh(gA(\tau)), g\bar{x}\sinh(gA(\tau))\right) \\
&= \left(\cosh\left(\frac{\tau}{\bar{x}}\right), \sinh\left(\frac{\tau}{\bar{x}}\right)\right), \qquad (5)
\end{aligned}$$

where we have employed the coordinate transformation between the barred and un-barred coordinates derived last week.

In the un-barred frame, we have

$$a^{\mu} \doteq \frac{d\dot{\xi}^{\mu}}{d\tau} = \frac{1}{\bar{x}}\left(\sinh\left(\frac{\tau}{\bar{x}}\right), \cosh\left(\frac{\tau}{\bar{x}}\right)\right). \qquad (6)$$

Note that

$$\sqrt{a^{\mu}a_{\mu}} = \frac{1}{\bar{x}}, \qquad (7)$$

as it should.

[Note, you might wonder why we did not need to worry about the connection to compute the 4-velocity in the first place. This is because $\xi^{\bar{\mu}}(\tau)$ isn't itself a four-vector; it is just a series of spacetime events, labeled with the parameter τ, which define the worldline of this observer. The tangent to this trajectory, $d\xi^{\bar{\mu}}/d\tau$, is the 4-velocity.]

(b) [5 pts] On a spacetime diagram, show the trajectory (t, x) exhibited by the uniformly accelerated astronaut from Problem 6 of pset 2.

Solution: See Figure 1. Recall that the astronaut's trajectory was given by

$$(t, x) = \frac{1}{g}\left(\sinh(g\bar{t}), \cosh(g\bar{t})\right). \qquad (8)$$

(c) [5 pts] Show that there is a region of spacetime which is *causally disconnected* from this astronaut. In other words, show that there is a region of spacetime in which events cannot effect the astronaut without violating the fundamental postulate that information cannot propagate faster than the speed of light.

Solution: Light travels at unit slope in our diagram. It is clear that the trajectory of the astronaut is asymptoting to $t = x$ (because $\sinh(\tau) \to \cosh(\tau)$ as $\tau \to \infty$). Therefore, any light ray emitted from the region to the left of the line in Figure 2 emanating from the origin will never intersect the astronaut's trajectory. This region is causally disconnected from the astronaut.

(d) [5 pts] Find the boundary between the region that is causally connected and causally disconnected from the astronaut. Such a boundary is called a

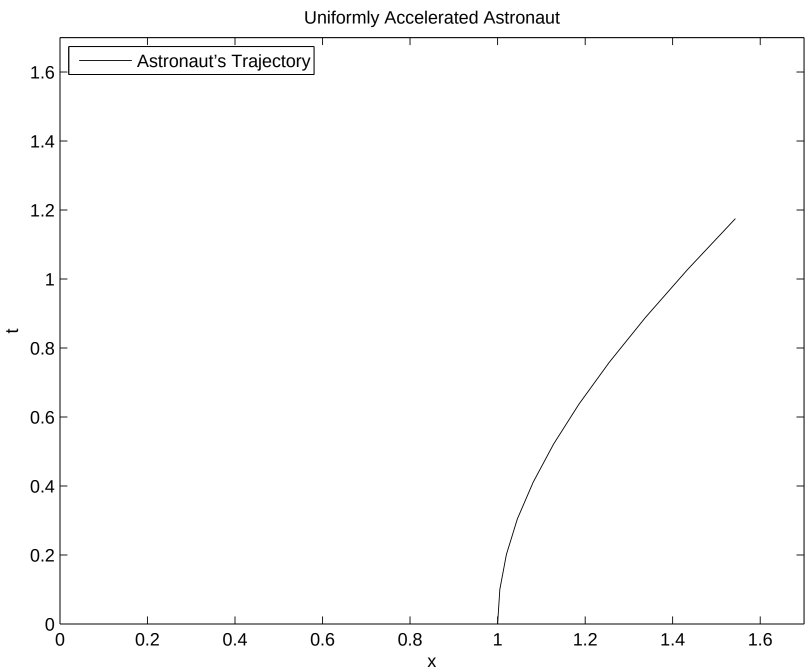

Figure 1: The trajectory of the astronaut from Problem 6 last week.

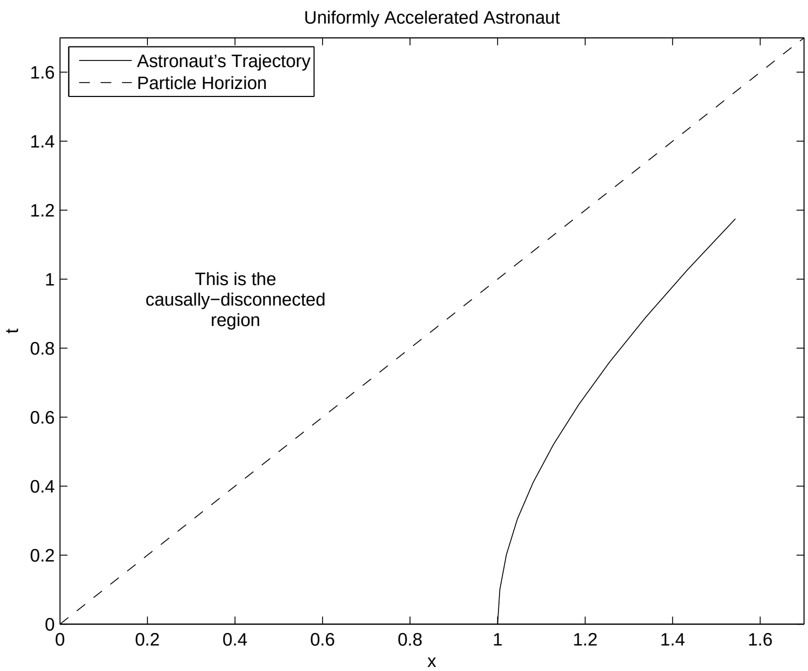

Figure 2: The trajectory of the astronaut, including a hypothetical light ray emitted from the origin in the (t, x) plane. Since the trajectory of the astronaut is asymptoting to the light ray, any light emitted from the region to the left of the ray will never intersect the astronaut's trajectory.

particle horizon; it shares some features of the *event horizon* that separates the interior and exterior spacetimes of black holes.

Throughout this problem, only consider $t \geq 0$.

Solution: The line emanating from the origin in Figure 2 is the particle horizon.

2. [5 pts] Perfect fluids:

In class, I listed one of the defining characteristics of a perfect fluid that it have no viscosity — i.e., no force parallel to the interface between fluid elements. This implied that the stress-energy tensor must be diagonal — any component T^{ij} for $i \neq j$ would violate this assumption. I then claimed that the stress-energy tensor could be written

$$T^{\alpha\beta} \doteq \text{diag}[\rho, P, P, P] \tag{9}$$

in Cartesian coordinates (t, x, y, z).

Suppose that the form were instead

$$T^{\alpha\beta} \doteq \text{diag}[\rho, P(1+\epsilon), P, P) \ . \tag{10}$$

Show that if one performs a rotation around the z axis by an angle ϕ that $T^{\alpha'\beta'}$ has off-diagonal components of order ϵP. Hence we must have $\epsilon = 0$ in order for the tensor to be diagonal in *all* Cartesian coordinate systems.

Solution: A rotation about the z axis is a coordinate transformation with the Jacobian

$$J^{\bar{\mu}}{}_{\mu} \equiv \frac{\partial x^{\bar{\mu}}}{\partial x^{\mu}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{11}$$

To transform the stress-energy tensor, we use

$$T_{\bar{\mu}\bar{\nu}} = \left(J^{-1}\right)^{\mu}{}_{\bar{\mu}} \left(J^{-1}\right)^{\nu}{}_{\bar{\nu}} T_{\mu\nu} = \left(J^{-1}\right)^{T} T J^{-1}, \tag{12}$$

which we see generates off-diagonal components of order $\epsilon P \sin\phi \cos\phi$.

3. "3+1" split of the electromagnetic field:

An observer with 4-velocity $\vec{U}$ interacting with an electromagnetic field $\mathbf{F}$ measures electric and magnetic fields $\vec{E}_{\vec{U}}$ and $\vec{B}_{\vec{U}}$ in their instantaneous local inertial reference frame (that is, in an orthonormal basis with $\vec{e}_{\hat{0}} = \vec{U}$). These fields are 4-vectors with components

$$E^{\alpha}_{\vec{U}} = F^{\alpha\beta} U_{\beta} \ , \qquad B^{\alpha}_{\vec{U}} = -\frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} U_{\beta} F_{\gamma\delta} \ . \tag{13}$$

(a) [7 pts] Show that $\vec{E}_{\vec{U}}$ and $\vec{B}_{\vec{U}}$ lie orthogonal to the observer's worldline. Thus, they are spatial vectors according to the observer, living entirely in that observer's hypersurface of simultaneity. (Hint: recall the projection tensor defined in Pset 1.)

Solution: Consider $\vec{E}_{\vec{U}} \cdot \vec{U}$:

$$E^{\alpha}_{\vec{U}} U_{\alpha} = F^{\alpha\beta} U_{\beta} U_{\alpha} = 0, \tag{14}$$

because F is antisymmetric and $U \otimes U$ is symmetric. Similarly, we have

$$B^{\alpha}_{\vec{U}} U_{\alpha} = -\frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} U_{\beta} F_{\gamma\delta} U_{\alpha} = 0. \tag{15}$$

Thus, $\vec{E}_{\vec{U}}$ and $\vec{B}_{\vec{U}}$ are orthogonal to $\vec{U}$. Because $\vec{U}$ is timelike, this implies that $\vec{E}_{\vec{U}}$ and $\vec{B}_{\vec{U}}$ are spacelike, and live in the observer's hypersurface of simultaneity.

(b) [8 pts] Show that the field tensor can be reconstructed from the observer's 4-velocity and the electric and magnetic fields they measure via the following tensor equation (valid for any basis):

$$F^{\alpha\beta} = U^{\alpha} E^{\beta}_{\vec{U}} - E^{\alpha}_{\vec{U}} U^{\beta} + \epsilon^{\alpha\beta}{}_{\gamma\delta} U^{\gamma} B^{\delta}_{\vec{U}} \,. \tag{16}$$

The identity

$$\begin{aligned} \epsilon_{\alpha\beta\gamma\chi} \epsilon^{\chi\rho\sigma\kappa} = \delta^{\rho}{}_{\alpha} \delta^{\sigma}{}_{\beta} \delta^{\kappa}{}_{\gamma} + \delta^{\sigma}{}_{\alpha} \delta^{\kappa}{}_{\beta} \delta^{\rho}{}_{\gamma} + \delta^{\kappa}{}_{\alpha} \delta^{\rho}{}_{\beta} \delta^{\sigma}{}_{\gamma} \\ - \delta^{\rho}{}_{\alpha} \delta^{\kappa}{}_{\beta} \delta^{\sigma}{}_{\gamma} - \delta^{\kappa}{}_{\alpha} \delta^{\sigma}{}_{\beta} \delta^{\rho}{}_{\gamma} - \delta^{\sigma}{}_{\alpha} \delta^{\rho}{}_{\beta} \delta^{\kappa}{}_{\gamma} \end{aligned} \tag{17}$$

may prove useful.

Solution: As an aside, generating these identities is a nice party trick (assuming you attend particularly nerdy parties). The important things to remember (so you don't have to look them up) are:

(a) The result must be antisymmetric in the free indices of each of the ϵ terms.

(b) The properties of the ϵ tensor density require that each of the free indices of the left-hand term equals one of the free indices of the right-hand term. (This means we can write the result in terms of Kroniker δs between the free indices.)

(c) There will be an overall multiplicative factor for the number of different ways to assign the same numbers to the set of contracted indices (for the hint, that is one—if we were contracting on two indices the factor would be two, if three indices, the factor would be 6, etc).

With that identity out of the way, consider the quantity

$$U^{\alpha} E^{\beta}_{\vec{U}} - E^{\alpha}_{\vec{U}} U^{\beta} + \epsilon^{\alpha\beta}{}_{\gamma\delta} U^{\gamma} B^{\delta}_{\vec{U}}. \tag{18}$$

The part proportional to $\vec{E}_{\vec{U}}$ reduces to

$$U^\alpha E^\beta_{\vec{U}} - E^\alpha_{\vec{U}} U^\beta = U^\alpha F^\beta{}_\gamma U^\gamma - F^\alpha{}_\gamma U^\gamma U^\beta. \tag{19}$$

The part proportional to $\vec{B}_{\vec{U}}$ reduces to (for convenience, we're writing it with indices down for the moment)

$$\epsilon_{\alpha\beta\gamma\delta} U^\gamma B^\delta_{\vec{U}} = -\frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}\epsilon^{\delta\rho\kappa\sigma} U^\gamma U_\rho F_{\kappa\sigma}. \tag{20}$$

Applying the hint and exploiting the antisymmetry of F, we find

$$\epsilon_{\alpha\beta\gamma\delta} U^\gamma B^\delta_{\vec{U}} = -\left(\delta^\rho{}_\alpha \delta^\kappa{}_\beta \delta^\sigma{}_\gamma + \delta^\kappa{}_\alpha \delta^\sigma{}_\beta \delta^\rho{}_\gamma + \delta^\sigma{}_\alpha \delta^\rho{}_\beta \delta^\kappa{}_\gamma\right) U^\gamma U_\rho F_{\kappa\sigma}. \tag{21}$$

Simplifying the Kroniker δs yields

$$\epsilon_{\alpha\beta\gamma\delta} U^\gamma B^\delta_{\vec{U}} = -\left(U_\alpha U^\sigma F_{\beta\sigma} + U_\gamma U^\gamma F_{\alpha\beta} + U_\beta U^\kappa F_{\kappa\alpha}\right). \tag{22}$$

We see that the first and third terms in this expression are equal to the two terms in Equation (19) (once we re-raise the indices); recalling that $\vec{U}\cdot\vec{U} = -1$, we have

$$U^\alpha E^\beta_{\vec{U}} - E^\alpha_{\vec{U}} U^\beta + \epsilon^{\alpha\beta}{}_{\gamma\delta} U^\gamma B^\delta_{\vec{U}} = F^{\alpha\beta}, \tag{23}$$

which is what we were asked to show.

(c) [5 pts] The wedge product between two 1-forms is defined as

$$\tilde{A}\wedge\tilde{B} = \tilde{A}\otimes\tilde{B} - \tilde{B}\otimes\tilde{A}\,.$$

Although it is something of an abuse of this terminology, we define the wedge product between two vectors similarly:

$$\vec{A}\wedge\vec{B} = \vec{A}\otimes\vec{B} - \vec{B}\otimes\vec{A}\,.$$

(This generalization may seem obvious, but I am actually being somewhat cavalier about the notation — the wedge product, strictly speaking, is only defined for differential forms.)

The Hodge dual of a $(0,2)$ tensor is defined as

$${}^*C_{\mu\nu} = \frac{1}{2}\epsilon^{\alpha\beta}{}_{\mu\nu} C_{\alpha\beta}\,.$$

Show that the field tensor may be written

$$\mathbf{F} = a\,\vec{U}\wedge\vec{E}_{\vec{U}} + b\,{}^*\!\left(\vec{U}\wedge\vec{B}_{\vec{U}}\right)\,.$$

What are the values of the real constants a and b?

Additional (and more rigorous) discussion of differential forms, Hodge duals, and associated mathematical notions can be found in Sec. 2.9 of Carroll.

Solution: Again, it will be more convenient to work with the indices on F down. We have

$$F_{\alpha\beta} = U_\alpha E_\beta - U_\beta E_\alpha + \epsilon_{\alpha\beta\gamma\kappa} U^\gamma B^\kappa. \tag{24}$$

By inspection, the part proportional to $\vec{E}_{\vec{U}}$ is $\vec{U} \wedge \vec{E}$, so we conclude that $a = 1$. What about the $\vec{B}_{\vec{U}}$ term? Because ϵ is already anti-symmetric, we lose nothing by anti-symmetrizing $U^\gamma B^\kappa$:

$$\epsilon_{\alpha\beta\gamma\kappa} U^\gamma B^\kappa = \frac{1}{2} \epsilon_{\alpha\beta\gamma\kappa} \left(U^\gamma B^\kappa - U^\kappa B^\gamma \right) = \left[^* \left(U \wedge B \right) \right]_{\alpha\beta}, \tag{25}$$

so we conclude $b = 1$, too. To summarize,

$$\mathbf{F} = \vec{U} \wedge \vec{E}_{\vec{U}} + {}^*\!\left(\vec{U} \wedge \vec{B}_{\vec{U}} \right) \tag{26}$$

4. Transformation of Christoffel symbols:

(a) [10 pts] Show that, under a coordinate transformation, the components of the Christoffel symbol transform as follows:

$$\Gamma^{\alpha'}{}_{\beta'\gamma'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x^{\beta'}} \frac{\partial x^\gamma}{\partial x^{\gamma'}} \Gamma^\alpha{}_{\beta\gamma} - \frac{\partial^2 x^{\alpha'}}{\partial x^\beta \, \partial x^\gamma} \frac{\partial x^\beta}{\partial x^{\beta'}} \frac{\partial x^\gamma}{\partial x^{\gamma'}} \tag{27}$$

Do this by considering the form of the Christoffel symbol in terms of derivatives of the metric.

Warning/hint: You may find at the end of your calculation that you have instead derived a rule that looks like

$$\Gamma^{\alpha'}{}_{\beta'\gamma'} = L^{\alpha'}{}_\alpha L^\beta{}_{\beta'} L^\gamma{}_{\gamma'} \Gamma^\alpha{}_{\beta\gamma} + L^\beta{}_{\beta'} L^{\alpha'}{}_\gamma \partial_\beta L^\gamma{}_{\gamma'} \tag{28}$$

where $L^{\alpha'}{}_\alpha = \partial x^{\alpha'} / \partial x^\alpha$. This may look wrong — the sign on the final term is incorrect. Inspecting your result closely, you'll see that the matrix being differentiated in the second term of what you have derived is not quite the same as it appears in the form we've asked you to find — the primed and unprimed indices are in opposite locations.

By noting that $L^\gamma{}_{\gamma'} L^{\alpha'}{}_\gamma = \delta^{\alpha'}{}_{\gamma'}$, you should be able to show that these two formulas are equivalent.

Solution: First, a brief digression: it is important, in problems like this, to be clear about what is really happening regarding space-time points, our coordinate labels for them, and the components of tensors (or tensor-like objects) in the various bases introduced by the coordinate components. In general, we're working at a fixed spacetime point, call it P. We have two coordinate systems, in which the coordinates of P differ:

$$\begin{aligned} P &\doteq \{x^{\bar{\mu}}\} &(29)\\ &\doteq \{x^\mu\}\,. &(30) \end{aligned}$$

These coordinate systems introduce different bases for the tangent space at P, so we have

$$T^{\bar{\mu}\cdots}{}_{\bar{\nu}\ldots}(P) \neq T^{\mu\cdots}{}_{\nu\ldots}(P), \tag{31}$$

where $T^{\mu\cdots}{}_{\nu\ldots}(P)$ and $T^{\bar{\mu}\cdots}{}_{\bar{\nu}\ldots}$ are the components of some tensor T at point P in the two coordinate systems. Just because the components are different, however, doesn't mean that the tensor is different—applying the tensor to the appropriate number of one-forms and vectors, we have

$$\begin{aligned} T(\omega,\ldots,v,\ldots)|_P &= T^{\bar{\mu}\cdots}{}_{\bar{\nu}\ldots}(P)\omega_{\bar{\mu}}(P)\ldots v^{\bar{\nu}}(P)\ldots \\ &= T^{\mu\cdots}{}_{\nu\ldots}(P)\omega_{\mu}(P)\ldots v^{\nu}(P)\ldots \end{aligned} \tag{32}$$

We know how to compute the components of tensors in the barred coordinate system in terms of the components in the un-barred system. We use the Jacobian, defined by

$$L^{\bar{\mu}}{}_{\mu} \equiv \frac{\partial x^{\bar{\mu}}}{\partial x^{\mu}}, \tag{33}$$

and its inverse, defined so that

$$\left(L^{-1}\right)^{\mu}{}_{\bar{\nu}} L^{\bar{\nu}}{}_{\sigma} = \delta^{\mu}{}_{\sigma}. \tag{34}$$

Note that this happens to be equal to

$$\left(L^{-1}\right)^{\mu}{}_{\bar{\mu}} = \frac{\partial x^{\mu}}{\partial x^{\bar{\mu}}}, \tag{35}$$

so we only need to have an expression for $x(\bar{x})$ *or* $\bar{x}(x)$ to differentiate—we can get the other Jacobian just by inverting a matrix rather than having to invert the coordinate transformation.

We nonetheless have to be a bit careful here. For example, the expression $\partial_{\alpha}\left(L^{-1}\right)^{\mu}{}_{\bar{\nu}}$ means "examine how the Jacobian from barred to unbarred coordinates varies with respect to the unbarred coordinates." Given that the coordinate transformation itself varies from point to point, we can see that this must be a non-trivial function. If we take relations like Eq. (35) too seriously, however, we might conclude that $\partial_{\alpha}\left(L^{-1}\right)^{\mu}{}_{\bar{\nu}} = \partial_{\alpha}\partial_{\bar{\nu}}x^{\mu} = \partial_{\bar{\nu}}\partial_{\alpha}x^{\mu} = \partial_{\bar{\nu}}\delta^{\mu}{}_{\alpha} = 0$. In fact, the swapping of derivatives that we did here was *not* correct: in the expression $\partial_{\alpha}\partial_{\bar{\nu}}x^{\mu}$, x^{μ} was shorthand for the *function* $x^{\mu}(x^{\bar{\nu}})$; in the expression $\partial_{\bar{\nu}}\partial_{\alpha}x^{\mu}$, x^{μ} stood for a particular coordinate. Hence, we made an error in that step.

The transformed tensor components work out to

$$T^{\bar{\mu}\cdots}{}_{\bar{\nu}\ldots} = T^{\mu\cdots}{}_{\nu\ldots}L^{\bar{\mu}}{}_{\mu}\ldots\left(L^{-1}\right)^{\nu}{}_{\bar{\nu}}\cdots, \tag{36}$$

with everything evaluated at point P.

Finally, we have the operator identity that

$$\partial_{\bar{\mu}} = \left(L^{-1}\right)^{\mu}{}_{\bar{\mu}}\partial_{\mu}, \tag{37}$$

where the left-hand-side operates on functions of $\bar{x}$ at point P, and the right-hand-side operates on functions of x at point P.

We now return to our regularly scheduled programming....

The Christoffel symbol is given by

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\kappa} \left[\partial_\beta g_{\gamma\kappa} + \partial_\gamma g_{\beta\kappa} - \partial_\kappa g_{\beta\gamma} \right]. \tag{38}$$

We don't know (yet) how this transforms in a new coordinate system, but we can re-compute it (at the same spacetime point) using the metric in the new system:

$$\Gamma^{\bar{\alpha}}{}_{\bar{\beta}\bar{\gamma}} = \frac{1}{2} g^{\bar{\alpha}\bar{\bar{\kappa}}} \left[\partial_{\bar{\beta}} g_{\bar{\gamma}\bar{\kappa}} + \partial_{\bar{\gamma}} g_{\bar{\beta}\bar{\kappa}} - \partial_{\bar{\kappa}} g_{\bar{\beta}\bar{\gamma}} \right]. \tag{39}$$

Now we take a stiff drink, and re-write this in terms of the metric and derivatives in the un-barred coordinate system:

$$\begin{aligned}
\Gamma^{\bar{\alpha}}{}_{\bar{\beta}\bar{\gamma}} = \frac{1}{2} L^{\bar{\alpha}}{}_{\alpha} L^{\bar{\kappa}}{}_{\kappa_1} g^{\alpha\kappa_1} & \\
\times \Big[\left(L^{-1}\right)^{\beta}{}_{\bar{\beta}} \partial_\beta & \left(\left(L^{-1}\right)^{\gamma}{}_{\bar{\gamma}} \left(L^{-1}\right)^{\kappa_2}{}_{\bar{\kappa}} g_{\gamma\kappa_2} \right) \\
+ \left(L^{-1}\right)^{\gamma}{}_{\bar{\gamma}} \partial_\gamma & \left(\left(L^{-1}\right)^{\beta}{}_{\bar{\beta}} \left(L^{-1}\right)^{\kappa_2}{}_{\bar{\kappa}} g_{\beta\kappa_2} \right) \\
& - \left(L^{-1}\right)^{\kappa_2}{}_{\bar{\kappa}} \partial_{\kappa_2} \left(\left(L^{-1}\right)^{\beta}{}_{\bar{\beta}} \left(L^{-1}\right)^{\gamma}{}_{\bar{\gamma}} g_{\beta\gamma} \right) \Big].
\end{aligned} \tag{40}$$

We can expand the derivatives of products using the product rule. Note that the terms with derivatives of g (only) just reproduce terms which amount to transforming Γ as a tensor. We can move these uninteresting terms over to the left, and define

$$\Delta\Gamma^{\bar{\alpha}}{}_{\bar{\beta}\bar{\gamma}} \equiv \Gamma^{\bar{\alpha}}{}_{\bar{\beta}\bar{\gamma}} - L^{\bar{\alpha}}{}_{\alpha} \left(L^{-1}\right)^{\beta}{}_{\bar{\beta}} \left(L^{-1}\right)^{\gamma}{}_{\bar{\gamma}} \Gamma^{\alpha}{}_{\beta\gamma}, \tag{41}$$

which represents the "non-tensorial" parts of the Γ transformation.

We have

$$\begin{aligned}
\Delta\Gamma^{\bar{\alpha}}{}_{\bar{\beta}\bar{\gamma}} = \frac{1}{2} L^{\bar{\alpha}}{}_{\alpha} L^{\bar{\kappa}}{}_{\kappa_1} g^{\alpha\kappa_1} & \\
\times \Big[\left(L^{-1}\right)^{\beta}{}_{\bar{\beta}} g_{\gamma\kappa_2} \partial_\beta & \left(\left(L^{-1}\right)^{\gamma}{}_{\bar{\gamma}} \left(L^{-1}\right)^{\kappa_2}{}_{\bar{\kappa}} \right) \\
+ \left(L^{-1}\right)^{\gamma}{}_{\bar{\gamma}} g_{\beta\kappa_2} \partial_\gamma & \left(\left(L^{-1}\right)^{\beta}{}_{\bar{\beta}} \left(L^{-1}\right)^{\kappa_2}{}_{\bar{\kappa}} \right) \\
& - \left(L^{-1}\right)^{\kappa_2}{}_{\bar{\kappa}} g_{\beta\gamma} \partial_{\kappa_2} \left(\left(L^{-1}\right)^{\beta}{}_{\bar{\beta}} \left(L^{-1}\right)^{\gamma}{}_{\bar{\gamma}} \right) \Big].
\end{aligned} \tag{42}$$

The symmetry properties of Γ imply that we can collapse the first two terms, and that the last term expands to the expression below:

$$\Delta\Gamma^{\bar{\alpha}}{}_{\bar{\beta}\bar{\gamma}} = \frac{1}{2} L^{\bar{\alpha}}{}_{\alpha} L^{\bar{\kappa}}{}_{\kappa_1} g^{\alpha\kappa_1} \times \Big[2\big(L^{-1}\big)^{\beta}{}_{\bar{\beta}} g_{\gamma\kappa_2} \partial_\beta \Big(\big(L^{-1}\big)^{\gamma}{}_{\bar{\gamma}} \big(L^{-1}\big)^{\kappa_2}{}_{\bar{\kappa}} \Big) - 2\big(L^{-1}\big)^{\kappa_2}{}_{\bar{\kappa}} g_{\beta\gamma} \big(L^{-1}\big)^{\beta}{}_{\bar{\beta}} \partial_{\kappa_2} \big(L^{-1}\big)^{\gamma}{}_{\bar{\gamma}} \Big]. \quad (43)$$

Now we expand the derivative in the first term in brackets, and exploit the cancellation of J and J^{-1}, to obtain

$$\Delta\Gamma^{\bar{\alpha}}{}_{\bar{\beta}\bar{\gamma}} = L^{\bar{\alpha}}{}_{\alpha} \Big[\big(L^{-1}\big)^{\beta}{}_{\bar{\beta}} \partial_\beta \big(L^{-1}\big)^{\alpha}{}_{\bar{\gamma}} + \big(L^{-1}\big)^{\beta}{}_{\bar{\beta}} \big(L^{-1}\big)^{\gamma}{}_{\bar{\gamma}} g_{\gamma\kappa_2} g^{\alpha\kappa_1} L^{\bar{\kappa}}{}_{\kappa_1} \partial_\beta \big(L^{-1}\big)^{\kappa_2}{}_{\bar{\kappa}} - g^{\alpha\kappa_1} g_{\beta\gamma} \big(L^{-1}\big)^{\beta}{}_{\bar{\beta}} \partial_{\kappa_1} \big(L^{-1}\big)^{\gamma}{}_{\bar{\gamma}} \Big]. \quad (44)$$

Note that

$$\big(L^{-1}\big)^{\beta}{}_{\bar{\gamma}} \partial_\beta \big(L^{-1}\big)^{\alpha}{}_{\bar{\rho}} = \partial_{\bar{\gamma}} \partial_{\bar{\rho}} x^\alpha \quad (45)$$

$$= \partial_{\bar{\rho}} \partial_{\bar{\gamma}} x^\alpha \quad (46)$$

$$= \big(L^{-1}\big)^{\rho}{}_{\bar{\rho}} \partial_\rho \big(L^{-1}\big)^{\alpha}{}_{\bar{\gamma}}. \quad (47)$$

(The indices $\bar{\gamma}$ and $\bar{\beta}$ have traded places.) Applying this to the middle term in brackets of Equation (44), we get

$$\Delta\Gamma^{\bar{\alpha}}{}_{\bar{\beta}\bar{\gamma}} = L^{\bar{\alpha}}{}_{\alpha} \Big[\big(L^{-1}\big)^{\beta}{}_{\bar{\beta}} \partial_\beta \big(L^{-1}\big)^{\alpha}{}_{\bar{\gamma}} + \big(L^{-1}\big)^{\beta}{}_{\bar{\kappa}} \big(L^{-1}\big)^{\gamma}{}_{\bar{\gamma}} g_{\gamma\kappa_2} g^{\alpha\kappa_1} L^{\bar{\kappa}}{}_{\kappa_1} \partial_\beta \big(L^{-1}\big)^{\kappa_2}{}_{\bar{\beta}} - g^{\alpha\kappa_1} g_{\beta\gamma} \big(L^{-1}\big)^{\beta}{}_{\bar{\beta}} \partial_{\kappa_1} \big(L^{-1}\big)^{\gamma}{}_{\bar{\gamma}} \Big], \quad (48)$$

which can be reduced to

$$\Delta\Gamma^{\bar{\alpha}}{}_{\bar{\beta}\bar{\gamma}} = L^{\bar{\alpha}}{}_{\alpha} \Big[\big(L^{-1}\big)^{\beta}{}_{\bar{\beta}} \partial_\beta \big(L^{-1}\big)^{\alpha}{}_{\bar{\gamma}} + \big(L^{-1}\big)^{\gamma}{}_{\bar{\gamma}} g_{\gamma\kappa_2} g^{\alpha\kappa_1} \partial_{\kappa_1} \big(L^{-1}\big)^{\kappa_2}{}_{\bar{\beta}} - g^{\alpha\kappa_1} g_{\beta\gamma} \big(L^{-1}\big)^{\beta}{}_{\bar{\beta}} \partial_{\kappa_1} \big(L^{-1}\big)^{\gamma}{}_{\bar{\gamma}} \Big]. \quad (49)$$

We see that the last two terms in brackets cancel, leaving us with

$$\Delta\Gamma^{\bar{\alpha}}{}_{\bar{\beta}\bar{\gamma}} = L^{\bar{\alpha}}{}_{\alpha} \big(L^{-1}\big)^{\beta}{}_{\bar{\beta}} \partial_\beta \big(L^{-1}\big)^{\alpha}{}_{\bar{\gamma}}. \quad (50)$$

But, we know that

$$\partial_\beta \Big(\big(L^{-1}\big)^{\alpha}{}_{\bar{\gamma}} L^{\bar{\alpha}}{}_{\alpha} \Big) = 0, \quad (51)$$

so we can swap the Jacobian outside the derivative with the inverse Jacobian inside the derivative at the cost of a minus sign:

$$\Delta\Gamma^{\bar{\alpha}}{}_{\bar{\beta}\bar{\gamma}} = -\left(L^{-1}\right)^{\alpha}{}_{\bar{\gamma}}\left(L^{-1}\right)^{\beta}{}_{\bar{\beta}}\partial_{\beta}L^{\bar{\alpha}}{}_{\alpha} = -\frac{\partial x^{\alpha}}{\partial x^{\bar{\gamma}}}\frac{\partial x^{\beta}}{\partial x^{\bar{\beta}}}\frac{\partial^2 x^{\bar{\alpha}}}{\partial x^{\beta}\partial x^{\alpha}}, \tag{52}$$

which is what we were asked to show.

(b) [5 pts] Show that, using this rule, the components of the covariant derivative of a vector transform as tensors should:

$$\nabla_{\alpha'}A^{\beta'} = \frac{\partial x^{\alpha}}{\partial x^{\alpha'}}\frac{\partial x^{\beta'}}{\partial x^{\beta}}\nabla_{\alpha}A^{\beta}\ . \tag{53}$$

Solution: We have

$$\nabla_{\alpha'}A^{\beta'} = \partial_{\alpha'}A^{\beta'} + \Gamma^{\beta'}{}_{\alpha'\gamma'}A^{\gamma'}; \tag{54}$$

we already computed the non-tensorial parts of the transformation of Γ in part (a) (see equation (52)), and now we will see that they exactly cancel the non-tensorial terms from the partial derivative above.

We have

$$\Delta\partial_{\alpha'}A^{\beta'} = \left(L^{-1}\right)^{\alpha}{}_{\alpha'}A^{\beta}\partial_{\alpha}L^{\beta'}{}_{\beta}, \tag{55}$$

where we are using the Δ notation for the non-tensorial terms as we did in part (a). But, we saw that

$$\Delta\Gamma^{\beta'}{}_{\alpha'\gamma'}A^{\gamma'} = -A^{\beta}\left(L^{-1}\right)^{\alpha}{}_{\alpha'}\partial_{\alpha}L^{\beta'}{}_{\beta}, \tag{56}$$

which is exactly what is required to cancel $\Delta\partial_{\alpha'}A^{\beta'}$, so $\nabla_{\alpha}A^{\beta}$ transforms like a tensor.

5. [12 pts] Carroll: Chapter 3, Problem 2. In this problem, define the curl via

$$(\mathbf{curl}\,\vec{V})_i = \epsilon_i{}^{jk}\nabla_j V_k\ . \tag{57}$$

(3 points for gradient, 4 points for divergence, 5 points for curl.)

Solution: First, as an overall comment, there's a bit of freedom here to interpret Carroll's question. We can imagine raising indices at the end of the problem to make things look vector-component-like; we can imagine projecting onto basis objects to construct a real vector (rather than just discussing the components of a vector); and we can imagine projecting onto an orthonormal basis to facilitate comparison with standard formulas in many other textbooks. We will grade this somewhat leniently as a consequence.

In Cartesian coordinates, the connection vanishes:

$$\Gamma^{\alpha}{}_{\beta\gamma} \doteq 0. \tag{58}$$

Using the formula for the transformation of the connection, we find that

$$\Gamma^{\alpha'}{}_{\beta'\gamma'} = -\left(L^{-1}\right)^{\gamma}{}_{\gamma'}\partial_{\beta'}L^{\alpha'}{}_{\gamma}, \tag{59}$$

where the "primed" coordinates are (r, θ, ϕ). In this case, it is most convenient to compute

$$\left(L^{-1}\right)^{\alpha}{}_{\alpha'} \equiv \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)}, \tag{60}$$

since we can more easily express (x, y, z) in terms of (r, θ, ϕ) than the inverse. To actually compute the quantities, we'll use Mathematica—see the pdf of the Mathematica worksheet in "pset03sol-5.pdf".

6. [12 pts] Carroll: Chapter 3, Problem 3. (3 points for each identity.)

Solution: For a diagonal metric, we have

$$g_{\mu\nu} = a_{\mu}\delta_{\mu\nu} \quad \text{and} \quad g^{\mu\nu} = \frac{1}{a_{\mu}}\delta^{\mu\nu} \tag{61}$$

(no summation implied by the repeated μ.) We have

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2}g^{\alpha\kappa}\left(\partial_{\beta}g_{\kappa\gamma} + \partial_{\gamma}g_{\kappa\beta} - \partial_{\kappa}g_{\beta\gamma}\right), \tag{62}$$

which reduces (using the above metrics, again without summation over repeated indices on a) to

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2}\left(\delta^{\alpha}{}_{\gamma}\partial_{\beta}\ln a_{\alpha} + \delta^{\alpha}{}_{\beta}\partial_{\gamma}\ln a_{\alpha} - \frac{1}{a_{\alpha}}\delta_{\beta\gamma}\partial_{\alpha}a_{\beta}\right). \tag{63}$$

If $\alpha \neq \beta \neq \gamma$ then all three terms vanish, so $\Gamma^{\alpha}{}_{\beta\gamma} = 0$, where we take Carroll's convention that different index labels imply that the indices are distinct. If $\beta = \gamma$, we get

$$\Gamma\alpha\beta\beta = -\frac{1}{2}\frac{1}{a_{\alpha}}\partial_{\alpha}a_{\beta}, \tag{64}$$

which is exactly what we find in Carroll.

We also have

$$\Gamma^{\alpha}{}_{\beta\alpha} = \frac{1}{2}\partial_{\beta}\ln a_{\alpha} = \partial_{\beta}\ln\sqrt{a_{\alpha}} \tag{65}$$

and

$$\Gamma^{\alpha}{}_{\alpha\alpha} = \partial_{\alpha}\ln a_{\alpha} - \frac{1}{2}\partial_{\alpha}\ln a_{\alpha} = \partial_{\alpha}\ln\sqrt{a_{\alpha}}, \tag{66}$$

also exactly what is in Carroll.

7. Prove the following connection identities:

(a) [2 pts] $\partial_\lambda g_{\mu\nu} = \Gamma_{\mu\nu\lambda} + \Gamma_{\nu\mu\lambda}$.

Solution: We know that we have a metric compatible connection:

$$\nabla_\mu g_{\nu\rho} = 0 = \partial_\mu g_{\nu\rho} - \Gamma^\alpha{}_{\mu\nu} g_{\alpha\rho} - \Gamma^\alpha{}_{\mu\rho} g_{\nu\alpha}. \tag{67}$$

Using the contracted metric to lower the indices on Γ, we obtain

$$\partial_\mu g_{\nu\rho} = \Gamma_{\rho\mu\nu} + \Gamma_{\nu\mu\rho}. \tag{68}$$

Exploiting the symmetry in the last two indices of Γ, we obtain the desired identity:

$$\partial_\lambda g_{\mu\nu} = \Gamma_{\mu\nu\lambda} + \Gamma\nu\mu\lambda. \tag{69}$$

(b) [3 pts] $g_{\mu\kappa}\partial_\lambda g^{\kappa\nu} = -g^{\kappa\nu}\partial_\lambda g_{\mu\kappa}$.

Solution: We have

$$\partial_\lambda \left(g_{\mu\kappa} g^{\kappa\nu}\right) = \partial_\lambda \delta^\nu{}_\mu = 0 = g_{\mu\kappa}\partial_\lambda g^{\kappa\nu} + g^{\kappa\nu}\partial_\lambda g_{\mu\kappa}. \tag{70}$$

(c) [3 pts] $\partial_\lambda g^{\mu\nu} = -\Gamma^\mu{}_{\lambda\kappa} g^{\kappa\nu} - \Gamma^\nu{}_{\lambda\kappa} g^{\kappa\mu}$

Solution: If the metric is covariantly conserved, so is the inverse metric:

$$\nabla_\lambda g^{\mu\nu} = 0 = \partial_\lambda g^{\mu\nu} + \Gamma^\mu{}_{\lambda\alpha} g^{\alpha\nu} + \Gamma^\nu{}_{\lambda\alpha} g^{\mu\alpha} \tag{71}$$

The next three parts rely on an identity I plan to prove in lecture on Tuesday March 4th. The quantity g is the determinant of the metric $g_{\mu\nu}$.

(d) [3 pts] $\nabla_\nu A_\mu{}^\nu = |g|^{-1/2}\partial_\nu\left(|g|^{1/2} A_\mu{}^\nu\right) - \Gamma^\lambda{}_{\nu\mu} A_\lambda{}^\nu$ in a coordinate basis.

Solution: Hopefully, Scott proved that

$$\Gamma^\mu{}_{\mu\lambda} = |g|^{-1/2}\partial_\lambda |g|^{1/2}. \tag{72}$$

Then, we have

$$\nabla_\nu A_\mu{}^\nu = \partial_\nu A_\mu{}^\nu + \Gamma^\nu{}_{\nu\alpha} A_\mu{}^\alpha - \Gamma^\alpha{}_{\nu\mu} A_\alpha{}^\nu = |g|^{-1/2}\partial_\nu \left(|g|^{1/2} A_\mu{}^\nu\right) - \Gamma^\alpha{}_{\nu\mu} A_\alpha{}^\nu \tag{73}$$

(e) [2 pts] $\nabla_\nu F^{\mu\nu} = |g|^{-1/2}\partial_\nu\left(|g|^{1/2} F^{\mu\nu}\right)$ in a coordinate basis, if $F^{\mu\nu}$ is antisymmetric.

Solution: We have

$$\nabla_\mu F^{\mu\nu} = \partial_\nu F^{\mu\nu} + \Gamma^\mu{}_{\nu\alpha} F^{\alpha\nu} + \Gamma^\nu{}_{\nu\alpha} F^{\mu\alpha}. \tag{74}$$

The second term vanishes because of the symmetry of Γ and the antisymmetry of F. The first and third can be combined into

$$\nabla_\mu F^{\mu\nu} = |g|^{-1/2}\partial_\mu \left(|g|^{1/2} F^{\mu\nu}\right) \tag{75}$$

(f) [3 pts] $\Box S \equiv g^{\mu\nu}\nabla_\mu \nabla_\nu S = |g|^{-1/2}\partial_\mu \left(|g|^{1/2} g^{\mu\nu}\partial_\nu S\right)$ in a coordinate basis. (S is a scalar function.)

Solution: We have

$$\Box S = g^{\mu\nu}\nabla_\mu \nabla_\nu S = \nabla_\mu \left(g^{\mu\nu}\partial_\mu S\right) = \partial_\mu \left(g^{\mu\nu}\partial_\nu S\right) + \Gamma^\mu{}_{\mu\alpha} g^{\alpha\nu}\partial_\nu S. \quad (76)$$

These terms can be combined into

$$|g|^{-1/2}\partial_\mu \left(|g|^{1/2} g^{\mu\nu}\partial_\nu S\right). \quad (77)$$