

8.962 Problem Set 4 Solutions

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1. [10 pts] Connection in Rindler spacetime

Recall that “Rindler spacetime”, which describes flat spacetime as observed by an accelerated observer, that we derived on Pset 2:

$$ds^2 = -(1 + g\bar{x})^2 d\bar{t}^2 + d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2 \quad (1)$$

(I’ve shifted the origin slightly, so that $g\bar{x} \rightarrow 1 + g\bar{x}$.) Compute all non-zero Christoffel symbols for this spacetime. (Carroll problem 3.3 will help you quite a bit here.)

Solution: We could use Carroll 3.3, but it’s often easier to just have the computer grind through the solution to problems like this (I always seem to mess up when trying to see which components of Γ are “obviously” zero). Besides, it’s good practice in developing Mathematica worksheets to do this sort of computation when things are a bit trickier. The attached worksheet (or PDF), called “pset04sol-notebook” demonstrates the calculation. The end result is that

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{g}{1 + g\bar{x}} \quad (2)$$

$$\Gamma^1_{00} = g(1 + g\bar{x}), \quad (3)$$

and all other components are zero.

2. Relativistic Euler equation

(a) [8 pts] Recall the stress-energy tensor for a perfect fluid, $T_{\alpha\beta} = \rho u_\alpha u_\beta + Ph_{\alpha\beta}$, where $h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$ (where u^α describes the 4-velocity of an element of the fluid). Use local energy momentum conservation, $\nabla_\alpha T^{\alpha\beta} = 0$, to derive the relativistic Euler equation,

$$(\rho + P)u^\alpha \nabla_\alpha u^\beta = -h^{\alpha\beta} \nabla_\alpha P. \quad (4)$$

Comment: If you just evaluate $\nabla_\alpha T^{\alpha\beta} = 0$, you will have a hard time getting a useful answer. You will get something that, technically is *an* Euler equation, but it won’t be one that reduces naturally to a form that is useful for most calculations. In particular, you won’t be able to obtain the non-relativistic limit in part (b).

A more useful form is obtained by separately equating to zero the components of $\nabla_\alpha T^{\alpha\beta}$ parallel to and orthogonal to the fluid's 4-velocity, u^α . Define $j^\beta = \nabla_\alpha T^{\alpha\beta}$. Then, the equations

$$\begin{aligned} j^\beta u_\beta &= 0 && \text{(Component parallel to } \vec{u}\text{)} \\ j^\beta h^\gamma{}_\beta &= 0 && \text{(Component orthogonal to } \vec{u}\text{)} \end{aligned}$$

give us useful information. The second equation in particular should provide us with a useful Euler equation. (Any idea what the first equation tells us?)

Solution: Since the fluid velocity, u^α , is a natural four-vector which appears in the physics of the problem, it makes sense to decompose all four-vectors into components along u^α and orthogonal to u^α . In the case of $j^\beta = \nabla_\alpha T^{\alpha\beta}$, which represents the four-vector flow of stress-energy into and out of a small volume about each point in spacetime, the parallel component will represent the stress-energy flux carried along the fluid, while the orthogonal components will represent the stress-energy carried by the squishing or expansion of neighboring fluid trajectories.

Writing out the expression for $h^\gamma{}_\beta j^\beta$, we can immediately drop any terms in j^β which are proportional to u^β , since these are projected to zero:

$$0 = h^\gamma{}_\beta j^\beta = h^\gamma{}_\beta \nabla_\alpha T^{\alpha\beta} \tag{5}$$

$$= h^\gamma{}_\beta [\nabla_\alpha (\rho u^\alpha u^\beta) + \nabla_\alpha (P h^{\alpha\beta})] \tag{6}$$

$$= h^\gamma{}_\beta \rho u^\alpha \nabla_\alpha u^\beta + h^{\gamma\alpha} \nabla_\alpha P + h^\gamma{}_\beta P u^\alpha \nabla_\alpha u^\beta. \tag{7}$$

This last can be written as

$$h^\gamma{}_\beta j^\beta = h^\gamma{}_\beta [(\rho + P) u^\alpha \nabla_\alpha u^\beta + h^{\beta\alpha} \nabla_\alpha P] = 0. \tag{8}$$

We know that $u^\alpha \nabla_\alpha u^\beta$ is orthogonal to u^β (this is just the acceleration along a worldline of a fluid element), and the second term in brackets is clearly orthogonal to u^β , so we can simply drop the projector $h^\gamma{}_\beta$, leaving

$$h^\gamma{}_\beta j^\beta = (\rho + P) u^\alpha \nabla_\alpha u^\gamma + h^{\gamma\alpha} \nabla_\alpha P = 0, \tag{9}$$

which is the Euler equation we were asked to derive.

The physical interpretation of the Euler equation is that the acceleration of a fluid element is proportional to the orthogonal gradient of the pressure, with proportionality constant $1/(\rho + P)$ —this is the relativistic generalization of “ $F = ma$ ” for fluids. We see that the quantity $\rho + P$ is playing the role of “mass” (more precisely, *inertial mass*). Note that equation (9), though it looks like a four-vector equation, has only three independent components (since it lives entirely in a subspace orthogonal to u^β).

As an aside, consider the component of j^β proportional to u^β , which we can extract by evaluating

$$0 = j^\beta u_\beta = u_\beta \nabla_\alpha T^{\alpha\beta} \quad (10)$$

$$= u_\beta [\nabla_\alpha (\rho u^\alpha u^\beta) + \nabla_\alpha (Ph^{\alpha\beta})] \quad (11)$$

$$= -\nabla_\alpha (\rho u^\alpha) - P \nabla_\alpha u^\alpha, \quad (12)$$

which implies

$$\nabla_\alpha (\rho u^\alpha) = -P \nabla_\alpha u^\alpha. \quad (13)$$

(Note that we have again exploited $u_\beta \nabla_\alpha u^\beta = 0$.)

Equation (13) has a simple interpretation, too. Recall that ρu^α is the energy flux of a perfect fluid. Then, equation (13) states that the flux of energy into a small region is opposite to the pressure times the divergence of fluid lines into that region. This is the relativistic version of “ $dE = -PdV$ ”, i.e. isentropic energy conservation.

(b) [4 pts] For a nonrelativistic fluid ($\rho \gg P$, $v^t \gg v^i$) and in a Cartesian basis, show that this equation reduces to the Euler equation,

$$\frac{\partial v_i}{\partial t} + v_k \partial_k v_i = -\frac{1}{\rho} \partial_i P. \quad (14)$$

(i, k are spatial indices running from 1 to 3.) What extra terms are present if the connection is non-zero (e.g., spherical coordinates)?

Solution: Since the velocity of the fluid is dominantly in the time direction, the orthogonal subspace consists of the three spatial directions. Thus, the Euler equation reduces to

$$\rho u^\alpha \nabla_\alpha u^i = -\nabla^i P, \quad (15)$$

where we have used that $P \ll \rho$ to reduce the coefficient of the left-hand side. Further specializing to a Cartesian coordinate system, and using that

$$u^\alpha \approx (1, \mathbf{v}), \quad (16)$$

we obtain

$$\frac{\partial v^i}{\partial t} + v^k \frac{\partial v^i}{\partial x^k} = -\frac{1}{\rho} \delta^{ik} \frac{\partial P}{\partial x^k}, \quad (17)$$

which is the equation we were asked to derive.

To determine what extra terms would be added in a non-Cartesian coordinate system, we return to equation (15) and insert the appropriate Γ terms:

$$\frac{\partial v^i}{\partial t} + v^k \frac{\partial v^i}{\partial x^k} + \Gamma_{00}^i + \Gamma_{0k}^i v^k + \Gamma_{jk}^i v^j v^k = -\frac{1}{\rho} g^{ik} \frac{\partial P}{\partial x^k}. \quad (18)$$

(c) [6 pts] Apply the relativistic Euler equation to Rindler spacetime for hydrostatic equilibrium. Hydrostatic equilibrium means that the fluid is

at rest in the \bar{x} coordinates, i.e. $u^{\bar{x}} = 0$. Suppose that the equation of state (relation between pressure and density) is $P = w\rho$ where w is a positive constant. Find the general solution $\rho(\bar{x})$ with $\rho(0) = \rho_0$.

Solution: If $u^{\bar{x}} = 0$, then we must have $u^{\bar{t}} = 1/(1 + g\bar{x})$ in order that $u^\mu u_\mu = -1$. We have

$$u^\alpha \nabla_\alpha u^\beta = u^{\bar{t}} \partial_{\bar{t}} u^\beta + u^{\bar{t}} \Gamma^{\bar{t}}_{\bar{t}\bar{t}} u^{\bar{t}}, \quad (19)$$

and u^β is independent of \bar{t} , so the left-hand side of the Euler equation becomes

$$\rho(1+w) u^\alpha \nabla_\alpha u^\beta = \rho(1+w) \Gamma^{\bar{t}}_{\bar{t}\bar{t}} u^{\bar{t}} u^{\bar{t}} = \rho(1+w) \frac{g}{1+g\bar{x}}. \quad (20)$$

(Note that the only the \bar{x} component is non-zero, as it should be, since the spatial dimension is clearly the orthogonal subspace to \vec{u} .)

The \bar{x} component of the right-hand side becomes

$$-h^{\bar{x}\beta} \nabla_\beta (w\rho) = -g^{\bar{x}\beta} \nabla_\beta (w\rho) = -w \partial_{\bar{x}} \rho. \quad (21)$$

Putting it together, we obtain

$$\partial_{\bar{x}} \ln \rho = -\frac{1+w}{w} \frac{g}{1+g\bar{x}}, \quad (22)$$

which has the solution

$$\rho(\bar{x}) = \rho_0 (1+g\bar{x})^{-(1+w)/w}. \quad (23)$$

(d) [6 pts] Suppose now instead that $w = w_0/(1+g\bar{x})$ where w_0 is a constant. Show that the solution is $\rho(\bar{x}) = \rho_0 \exp(-\bar{x}/L)$. Find L , the density scale height, in terms of g and w_0 . Convert to “normal” units by inserting appropriate factors of c — L should be a length.

Solution: We only need to modify equation (21)—the left-hand side of the Euler equation is unchanged. We now have

$$-h^{\bar{x}\beta} \nabla_\beta (w\rho) = w \partial_{\bar{x}} \rho + \rho \partial_{\bar{x}} w = -w \partial_{\bar{x}} \rho + \frac{g}{1+g\bar{x}} w \rho. \quad (24)$$

Borrowing the left-hand side we computed in the last part, we get

$$\partial_{\bar{x}} \ln \rho = -\frac{1}{w} \frac{g}{1+g\bar{x}} = -\frac{g}{w_0}. \quad (25)$$

The solution is

$$\rho(\bar{x}) = \rho_0 \exp\left(-\frac{g\bar{x}}{w_0}\right), \quad (26)$$

from which we see that

$$L = \frac{w_0}{g}. \quad (27)$$

Using the natural dimensions of P and ρ , we see that

$$[w_0] = \frac{(\text{length})^2}{(\text{time})^2}, \quad (28)$$

and, noting that $g\bar{x}$ must be dimensionless, we find that

$$[g] = \frac{1}{(\text{length})}, \quad (29)$$

so we have

$$L = \frac{w_0}{c^2 g}. \quad (30)$$

(e) [6 pts] Compare your solution to the density profile of a nonrelativistic, plane-parallel, isothermal atmosphere (for which $P = \rho kT/\mu$, where T is temperature and μ is the mean molecular weight) in a constant gravitational field. [Use the nonrelativistic Euler equation with gravity: add a term $-\partial_i\Phi = g_i$, where Φ is Newtonian gravitational potential and g_i is Newtonian gravitational acceleration, to the right hand side of Eq. (14).] Why does hydrostatic equilibrium in Rindler spacetime — where there is no gravity — give such similar results to hydrostatic equilibrium in a gravitational field?

Solution: The left-hand side of the Euler equation vanishes (because the atmosphere is at rest), and the right-hand side becomes:

$$0 = -\frac{kT}{\mu\rho}\partial_i\rho + g_i, \quad (31)$$

which has the solution

$$\rho = \rho_0 \exp\left(\frac{\mu g}{kT}x\right). \quad (32)$$

(Note that $g > 0$ implies gravitational force in the *positive* x direction, so the density is increasing in the direction of the force.)

That this is exactly the same form as the solution to the last part is not surprising because we know that constant acceleration in flat space (i.e. the Rindler situation) mimics the effects of a constant gravitational field.

3. [20 pts] Spherical hydrostatic equilibrium

As we shall derive later in the course, the line element for a spherically symmetric static spacetime may be written

$$ds^2 = -e^{2\Phi(r)}dt^2 + \left[1 - \frac{2GM(r)}{r}\right]^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where $\Phi(r)$ and $M(r)$ are some given functions. In hydrostatic equilibrium, the spatial components of the fluid's 4-velocity are all zero: $u^i = 0$

for $i \in [r, \theta, \phi]$. Using the relativistic Euler equation, show that in hydrostatic equilibrium $P = P(r)$ with

$$\frac{\partial P}{\partial r} = -(\rho + P) \frac{\partial \Phi}{\partial r}.$$

(Hint: Don't forget to enforce $\vec{u} \cdot \vec{u} = g_{\alpha\beta} u^\alpha u^\beta = -1$!)

Solution: Following the hint, note that $u^t = \exp(-\Phi(r))$ in order that $u^\mu u_\mu = -1$. Clearly, the orthogonal subspace to \vec{u} is the spatial dimensions. Because u^μ is time-independent and only $u^t \neq 0$, the left-hand side of the Euler equation reduces to

$$(\rho + P) u^\alpha \nabla_\alpha u^i = (\rho + P) \Gamma^i_{tt} u^t u^t. \quad (33)$$

Computing the connection (which we can do exactly as in problem 1—see “pset04sol-notebook” for details), we find that the only non-zero Γ^i_{tt} is

$$\Gamma^r_{tt} = e^{2\Phi(r)} \left(1 - \frac{2GM(r)}{r} \right) \Phi'(r), \quad (34)$$

so the r -component of the left-hand side becomes

$$(\rho + P) u^\alpha \nabla_\alpha u^r = \Gamma^r_{tt} u^t u^t = \left(1 - \frac{2GM(r)}{r} \right) \Phi'(r), \quad (35)$$

and the θ - and ϕ -components vanish. This implies that $\partial P / \partial \theta = \partial P / \partial \phi = 0$, so $P = P(r)$.

The r -component of the right-hand side is

$$-h^{r\beta} \nabla_\beta P = g^{rr} \partial_r P = - \left(1 - \frac{2GM(r)}{r} \right) \partial_r P. \quad (36)$$

Putting it together, we find

$$\frac{\partial P(r)}{\partial r} = -(\rho + P) \frac{\partial \Phi(r)}{\partial r}, \quad (37)$$

which is exactly what we were asked to show.

4. [20 pts] Converting from non-affine to affine parameterization
 Suppose $v^\alpha = dx^\alpha / d\lambda^*$ obeys the geodesic equation in the form

$$\frac{Dv^\alpha}{d\lambda^*} = \kappa(\lambda^*) v^\alpha.$$

Clearly λ^* is not an affine parameter.

Show that $u^\alpha = dx^\alpha / d\lambda$ obeys the geodesic equation in the form

$$\frac{Du^\alpha}{d\lambda} = 0$$

provided that

$$\frac{d\lambda}{d\lambda^*} = \exp \left[\int \kappa(\lambda^*) d\lambda^* \right] .$$

Solution: Consider a change of parameter, $\lambda^* \rightarrow \lambda(\lambda^*)$. Then

$$v^\alpha = \frac{dx^\alpha}{d\lambda^*} = \lambda' \frac{dx^\alpha}{d\lambda} = \lambda' u^\alpha, \quad (38)$$

where $\lambda' = d\lambda/d\lambda^*$. Also,

$$\frac{dv^\alpha}{d\lambda^*} = \lambda'' u^\alpha + (\lambda')^2 \frac{du^\alpha}{d\lambda}. \quad (39)$$

So, the left-hand side of the differential equation which v^α obeys becomes

$$\frac{Dv^\alpha}{d\lambda^*} = \frac{dv^\alpha}{d\lambda^*} + \Gamma^\alpha_{\beta\gamma} v^\beta v^\gamma = \lambda'' u^\alpha + (\lambda')^2 \frac{du^\alpha}{d\lambda} + \Gamma^\alpha_{\beta\gamma} (\lambda')^2 u^\beta u^\gamma, \quad (40)$$

and the right-hand side becomes

$$\kappa(\lambda^*) v^\alpha = \lambda' \kappa u^\alpha. \quad (41)$$

We see that u^α will satisfy the usual geodesic equation if

$$\lambda'' = \lambda' \kappa, \quad (42)$$

which implies that

$$\lambda'(\lambda^*) = \exp \left(\int^{\lambda^*} \kappa(s) ds \right), \quad (43)$$

which is what we were to show.

5. [20 pts] Conserved quantities with charge

A particle with electric charge e moves with 4-velocity u^α in a spacetime with metric $g_{\alpha\beta}$ in the presence of a vector potential A_μ . The equation describing this particle's motion can be written

$$u^\beta \nabla_\beta u_\alpha = e F_{\alpha\beta} u^\beta,$$

where

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha.$$

The spacetime admits a Killing vector field ξ^α such that

$$\begin{aligned} \mathcal{L}_\xi g_{\alpha\beta} &= 0, \\ \mathcal{L}_\xi A_\alpha &= 0. \end{aligned}$$

Show that the quantity $(u_\alpha + eA_\alpha)\xi^\alpha$ is constant along the worldline of the particle.

Solution: We will just compute by brute force the change in $(u_\alpha + eA_\alpha)\xi^\alpha$ along the particle's trajectory. (A more elegant solution would involve writing a Lagrangian whose Euler-Lagrange equation gives the equation of motion for the particle and applying Nöther's theorem, but this would require a lot of machinery to set up which we don't have yet.) We have

$$u^\beta \nabla_\beta [(u_\alpha + eA_\alpha)\xi^\alpha] = (u_\alpha + eA_\alpha)u^\beta \nabla_\beta \xi^\alpha + \xi^\alpha u^\beta \nabla_\beta (u_\alpha + eA_\alpha). \quad (44)$$

Note that, because $\mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$, we have

$$u^\alpha u^\beta \nabla_\beta \xi_\alpha = 0. \quad (45)$$

Applying this and the equation of motion for the particle, we have

$$u^\beta \nabla_\beta [(u_\alpha + eA_\alpha)\xi^\alpha] = eA_\alpha u^\beta \nabla_\beta \xi^\alpha + e\xi^\alpha u^\beta \nabla_\alpha A_\beta. \quad (46)$$

But, $\mathcal{L}_\xi A^\alpha = \xi^\beta \nabla_\beta A^\alpha - A^\beta \nabla_\beta \xi^\alpha = 0$, so we have

$$u^\beta \nabla_\beta [(u_\alpha + eA_\alpha)\xi^\alpha] = eA_\alpha u^\beta \nabla_\beta \xi^\alpha + eA^\alpha u^\beta \nabla_\alpha \xi_\beta. \quad (47)$$

Applying $\mathcal{L}_\xi g_{\mu\nu} = 0$ again, we obtain

$$u^\beta \nabla_\beta [(u_\alpha + eA_\alpha)\xi^\alpha] = 0, \quad (48)$$

so $(u_\alpha + eA_\alpha)\xi^\alpha$ is conserved along the particle's trajectory.