8.962 Pset 5 Solutions

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1. Space garbage

   In a convenient coordinate system, the spacetime of the earth is approximately

$$\begin{aligned} ds^2 &= -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1+\frac{2GM}{r}\right)\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] \\ &= -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1+\frac{2GM}{r}\right)\left(dx^2+dy^2+dz^2\right) , \qquad (1) \end{aligned}$$

   where M is the earth's mass. In the second version, we've remapped the spherical coordinates to Cartesian coordinates in the usual way:

$$x = \sin\theta\cos\phi\ , \qquad y = \sin\theta\sin\phi\ , \qquad z = \cos\theta\ . \qquad (2)$$

   Note that the Cartesian form of the spacetime metric is conveniently written $g_{\alpha\beta} = \eta_{\alpha\beta} - 2\Phi\text{diag}(1,1,1,1)$, with $\Phi \equiv -GM/r$. You may assume $|\Phi| \ll 1$ throughout this problem.

   The space shuttle orbits the earth in a circular ($u^r = 0$), equatorial ($\theta = \pi/2, u^\theta = 0$) orbit of radius R.

   (a) [5 pts] Using the geodesic equation, show that an orbit which begins equatorial remains equatorial: $du^\theta/dt = 0$ if $u^\theta = 0$ and $\theta = \pi/2$ at $t = 0$. (Hint: begin by computing the non-zero connection coefficients; use the fact that $\Phi \ll 1$ to simplify your answer. Use the results of Carroll 3.3.)

   Solution: We know from Carroll that, for a diagonal metric, the connection coefficients take the form

$$\begin{aligned} \Gamma^\alpha{}_{\beta\gamma} &= 0 && (3) \\ \Gamma^\alpha{}_{\alpha\beta} &= \partial_\beta \ln\sqrt{g_{\alpha\alpha}} && (4) \\ \Gamma^\alpha{}_{\beta\beta} &= -\frac{1}{2g_{\beta\beta}}\partial_\alpha g_{\beta\beta} && (5) \\ \Gamma^\alpha{}_{\alpha\alpha} &= \partial_\alpha \ln\sqrt{g_{\alpha\alpha}}, && (6) \end{aligned}$$

   where repeated indices are *not* summed over, and $\alpha \neq \beta \neq \gamma$. In this case, the non-constant part of the potential is Φ, which is already small, so all terms $(1/g)\partial g$ reduce to simply ∂g. In addition, $\partial_t \Phi = 0$, so we have, in the Cartesian coordinate system

$$\begin{aligned}
\Gamma^{\alpha}{}_{\alpha i} = \Gamma^{\alpha}{}_{i\alpha} &\doteq -\partial_i \Phi & (7)\\
\Gamma^{i}{}_{\beta\beta} &\doteq \partial_i \Phi & (8)\\
\Gamma^{i}{}_{ii} &\doteq -\partial_i \Phi. & (9)
\end{aligned}$$

Note that $\alpha \neq i$ and $\beta \neq i$ does not imply that α or β are timelike—they can be spacelike in a different dimension than i. In the spherical coordinate system the expressions are more complicated. Just for fun, we have explicitly computed the connection in spherical coordinates in the attached Mathematica notebook "pset05sol-notebook".

To demonstrate that an initially equatorial orbit remains equatorial (as it must by symmetry), we will have to use the geodesic equation. The θ component of the geodesic equation is

$$\frac{du^\theta}{d\tau} = -\Gamma^{\theta}{}_{\mu\nu} u^\mu u^\nu, \tag{10}$$

where τ is an affine parameter for the geodesic. Inserting the spherical coordinate form that we calculated, we find

$$\frac{du^\theta}{d\tau} = -\cos\theta \sin\theta (u^\phi)^2 \tag{11}$$

if we have $u^\theta = 0$. Evaluating at $\theta = \pi/2$, we find that $du^\theta/d\tau = 0$. Hence, if we start equatorial and our initial condition has $u^\theta = 0$, the orbit stays equatorial for all τ.

We now require that the orbit must remain circular: $du^r/d\tau = 0$.

(b) [5 pts] By enforcing this condition with the geodesic equation, derive an expression for the orbital frequency

$$\Omega \equiv \frac{d\phi/d\tau}{dt/d\tau}\ . \tag{12}$$

Does the result look familiar?

Solution: Writing the r component of the geodesic equation, we have

$$\frac{du^r}{d\tau} = 0 = -\Gamma^{r}{}_{\mu\nu} u^\mu u^\nu. \tag{13}$$

We specialize to an equatorial orbit ($\theta = \pi/2$, $u^\theta = 0$) and also require constant r (ie, $u^r = 0$). We explicitly compute this in the Mathematica notebook, and show that it leads to the result

$$\Omega \equiv \frac{u^\phi}{u^t} = \sqrt{\frac{GM}{r^3}} + \mathcal{O}(\epsilon^{3/2}) \tag{14}$$

where ϵ is the order counting parameter we suggested you include. The leading order term is just Kepler's law for a circular orbit.

The next part is most conveniently described in Cartesian coordinates; you may describe the shuttle's orbit as $x = R\cos\Omega t$, $y = R\sin\Omega t$.

An astronaut releases a bag of garbage into space, spatially displaced from the shuttle by $\xi^i = x^i_{\text{garbage}} - x^i_{\text{shuttle}}$.

(c) [10 pts] Using the equation of geodesic deviation, work out differential equations for the evolution of ξ^x, ξ^y, and ξ^z as a function of time. You may neglect terms in $(GM/r)^2$, and you may treat all orbital velocities as non-relativistic. You will need the Cartesian connection coefficients for this.

Solution: The geodesic deviation equation for ξ reads

$$\frac{D^2\xi^\mu}{d\tau^2} - R^\mu{}_{\nu\rho\kappa}u^\nu u^\rho \xi^\kappa = 0, \tag{15}$$

where τ is an affine parameter along the geodesic, u is the tangent vector to the geodesic, and

$$\frac{Dv^\mu}{d\tau} \equiv \frac{dv^\mu}{d\tau} + \Gamma^\mu{}_{\nu\rho}u^\nu v^\rho, \tag{16}$$

for any vector v defined on the geodesic. Assuming that all velocities in the problem are small, we can write

$$u^\mu \doteq \left(1 + \mathcal{O}(v^2), \mathcal{O}(\mathbf{v})\right), \tag{17}$$

where $\mathbf{v}$ is the spatial velocity for the orbit. Then, expanding the derivatives in equation (15), we have

$$\frac{d^2\xi^\mu}{d\tau^2} + 2\Gamma^\mu{}_{0\nu}\frac{d\xi^\nu}{d\tau} + \partial_0\left(\Gamma^\mu{}_{0\nu}\right)\xi^\nu + \Gamma^\mu{}_{0\kappa}\Gamma^\kappa{}_{0\nu}\xi^\nu - R^\mu{}_{00\kappa}\xi^\kappa = 0. \tag{18}$$

Because $\Gamma \sim \Phi \ll 1$ and $\partial_0\Gamma = 0$, this reduces to

$$\frac{d^2\xi^\mu}{d\tau^2} + 2\Gamma^\mu{}_{0\nu}\frac{d\xi^\nu}{d\tau} - R^\mu{}_{00\kappa}\xi^\kappa = 0. \tag{19}$$

Based on the answer to part a, in Cartesian coordinates we have

$$\Gamma^0{}_{0i} = \partial_i\Phi \tag{20}$$

$$\Gamma^i{}_{00} = -\partial_i\Phi \tag{21}$$

with all other $\Gamma^\mu{}_{0\nu} = 0$. Also, since any terms $\Gamma^2 \sim 0$, we have

$$R^\mu{}_{00\kappa} = \partial_0\Gamma^\mu{}_{0\kappa} - \partial_\kappa\Gamma^\mu{}_{00} = \begin{cases} 0 & \mu = 0 \text{ or } \kappa = 0 \\ -\partial_i\partial_j\Phi & \mu = i \text{ and } \kappa = j \end{cases}. \tag{22}$$

Plugging in, the t-component of the geodesic deviation equation becomes

$$\frac{d^2\xi^t}{d\tau^2} + 2\left(\partial_i \Phi\right)\frac{d\xi^i}{d\tau} = 0, \tag{23}$$

and the spatial components become

$$\frac{d^2\xi^i}{d\tau^2} - 2\left(\partial^i \Phi\right)\frac{d\xi^t}{d\tau} + \left(\partial^i\partial_j \Phi\right)\frac{d\xi^j}{d\tau} = 0. \tag{24}$$

These equations are displayed explicitly in the attached Mathematica worksheet. We see that there is a coupling between ξ^t, ξ^x, and ξ^y, but that the equation for ξ^z is independent of the others, since $\partial_z \Phi = 0$ in the equatorial plane.

(d) [5 pts] Suppose the initial displacement is $\xi^x = \xi^y = 0$, $\xi^z = L$, $d\xi^i/d\tau = 0$. Further, synchronize the clocks of the garbage and the space shuttle: $\xi^0 = 0$, $\partial_t \xi^0 = 0$. Has the astronaut succeeded in getting rid of the garbage?

Solution: If we also assume that $\xi^t = d\xi^t/dt = 0$ initially, then the equations we derived in the Mathematica worksheet give $\xi^t = \xi^x = \xi^y = 0$ for all time.

The z-equation is

$$\frac{d^2\xi^z}{dt^2} = -\frac{\Phi'(R)}{R}\xi^z = -\frac{GM}{R^3}\xi^z, \tag{25}$$

which, given the initial conditions above, has the solution

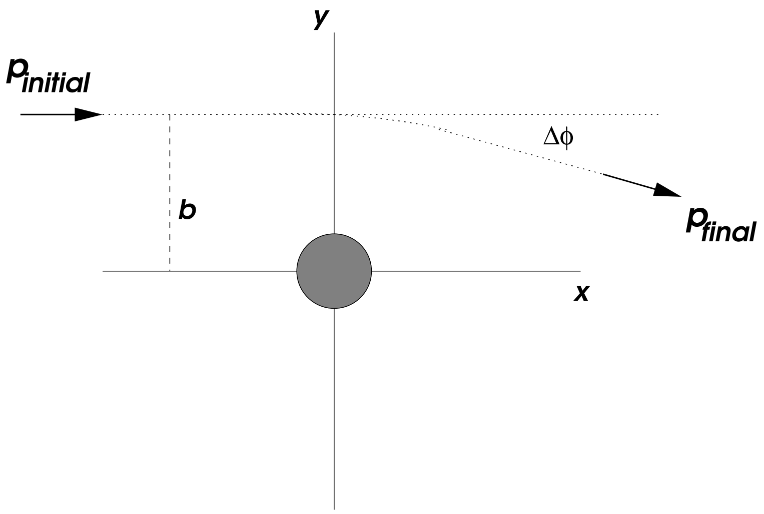
$$\xi^z(t) = L\cos\left(\sqrt{\frac{GM}{R^3}}t\right) = L\cos\Omega t. \tag{26}$$

Thus, the deviation vector between the astronaut and the garbage becomes zero twice an orbit, so the astronaut didn't succeed in getting rid of the garbage. From the point of view of the astronaut, the garbage executes harmonic oscillation in the vertical (z) direction with a period equal to the astronaut's orbital period.

2. [15 pts] Light bending

   The spacetime of the sun can be written using the same line element as that of Problem 1 (substituting the sun's mass for M). Consider a light ray which initially is moving purely along the x axis. Suppose that it passes near the sun with impact parameter (the distance between the axis passing through the sun's center and the axis along which the light initially moves) b, as shown in this figure:

   Using the geodesic equation, compute the angle by which the sun bends the light ray, $\Delta\phi = (p^y/p^x)_{\rm final}$, expressing your answer using GM and



b. You may assume that $GM/r \ll 1$ and that the bending angle is very small (so that $p^x_{\rm final} \simeq p^x_{\rm initial}$ to leading order in $\Delta\phi$).

Solution: We know that the light will follow a null geodesic trajectory:

$$\frac{du^\mu}{d\tau} = -\Gamma^\mu{}_{\nu\rho} u^\nu u^\rho, \tag{27}$$

where Γ is evaluated along the trajectory, and u is the tangent to the trajectory, i.e. $u^\mu = dx^\mu/d\tau$, with τ an affine parameter. We can write the solution to this equation as a series in Φ. At zero order in Φ, $\Gamma = 0$, and

$$u^{(0)\mu} \doteq (1, 1, 0, 0). \tag{28}$$

The first-order pieces of the geodesic equation are

$$\frac{du^{(1)\mu}}{d\tau} = -\left.\Gamma^{(1)\mu}{}_{\nu\rho}\right|_{x^{(0)}} u^{(0)\nu} u^{(0)\rho}, \tag{29}$$

where we can evaluate Γ on the zero-order trajectory because it is already first-order, so the error introduced by this is second-order in Φ.

We can compute the first-order change in u^y by integrating

$$\frac{du^{(1)y}}{d\tau} = u^{(0)x}\frac{du^{(1)y}}{dx} = -\left.\Gamma^y{}_{\mu\nu}\right|_{x^{(0)}} u^{(0)\mu} u^{(0)\nu} \tag{30}$$

along the initial trajectory from $x = -\infty$ to $x = \infty$. Explicitly evaluating the RHS of this equation in the attached Mathematica notebook gives

$$\frac{du^{(1)y}}{dx} = -\frac{2b\Phi'\left(\sqrt{x^2+b^2}\right)}{\sqrt{x^2+b^2}} = -\frac{2GMb}{(x^2+b^2)^{3/2}}. \tag{31}$$

Next, evaluate the integral:

$$\Delta u^{(1)y} = -\int_{-\infty}^{\infty} dx\,\frac{2GMb}{(x^2+b^2)^{3/2}} = -\frac{4GM}{b}. \tag{32}$$

The deflection angle, $\Delta\phi$ is then

$$\Delta\phi = -\frac{u^y}{u^x} \simeq \frac{4GM}{b} \tag{33}$$

using the small angle approximation.

3. [10 pts] Parallel transport on a sphere

   On the surface of a 2-sphere of radius a, $ds^2 = a^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$. Consider the vector $A_0 = \vec{e}_\theta$ at $\theta = \theta_0$, $\phi = 0$. The vector is parallel transported all the way around the latitude circle $\theta = \theta_0$ (i.e, over the range $0 \le \phi \le 2\pi$ at $\theta = \theta_0$). What is the resulting vector $\vec{A}$? What is its magnitude $(\vec{A}\cdot\vec{A})^{1/2}$? (Hint: derive differential equations for A^θ and A^ϕ as a function of ϕ.)

   Solution: The vector A will be parallel-transported around a curve with tangent vector v if

$$\frac{DA^i}{d\tau} = \frac{dA^i}{d\tau} + \Gamma^i{}_{jk} v^j A^k = 0, \tag{34}$$

or

$$\frac{dA^i}{d\tau} = -\Gamma^i{}_{jk} v^j A^k. \tag{35}$$

There is a condition on the tangent vector $v^i \equiv dx^i/d\tau$:

$$g_{ij} v^i v^j = \text{const} \tag{36}$$

on the curve. (This condition implies that τ is an affine parameter for the curve.)

We can parameterize the curve in question using $\tau = \phi$, which means that the tangent vector v is

$$v^i = \frac{dx^i}{d\tau} \doteq \frac{d}{d\phi}(\theta_0, \phi) = (0, 1). \tag{37}$$

We see that this parameterization is affine:

$$g_{ij} v^i v^j = a^2 \sin^2\theta_0 = \text{const}. \tag{38}$$

We could compute the connection coefficients exploiting that the metric is diagonal using the same formulas as in problem 1, but it's easier to just use Mathematica (see the attached worksheet). The result is

$$\Gamma^\theta{}_{\phi\phi} = -\cos\theta\sin\theta \tag{39}$$

$$\Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \cot\theta. \tag{40}$$

Thus, Eq. (35) reduces to the coupled ODEs

$$\frac{dA^\theta}{d\phi} = -\Gamma^\theta{}_{\phi\phi} A^\phi = \cos\theta_0 \sin\theta_0 A^\phi \tag{41}$$

and

$$\frac{dA^\phi}{d\phi} = -\Gamma^\phi{}_{\phi\theta} A^\theta = -\cot\theta_0 A^\theta . \tag{42}$$

By taking one additional derivative, we decouple them:

$$\frac{d^2A^\theta}{d\phi^2} = -\cos^2\theta_0 A^\theta \ , \tag{43}$$

$$\frac{d^2A^\phi}{d\phi^2} = -\cos^2\theta_0 A^\phi \ . \tag{44}$$

The solution to these equations, with $A^i(\phi = 0) \doteq (1, 0)$ is

$$A^\theta = \cos\left[\cos\theta_0 \phi\right] \ , \tag{45}$$

$$A^\phi = -\csc\theta_0 \sin\left[\cos\theta_0 \phi\right] \ . \tag{46}$$

When $\phi = 2\pi$, this reduces to

$$A^\theta = \cos\left[2\pi\cos\theta_0\right] \ , \tag{47}$$

$$A^\phi = -\csc\theta_0 \sin\left[2\pi\cos\theta_0\right] \ . \tag{48}$$

From either the solution at $\phi = 2\pi$ or from the general solution, it is simple to see that

$$g_{ij}A^iA^j = a. \tag{49}$$

This is implied by the rules of parallel transport since parallel transport doesn't change the magnitude of a vector.

4. Curvature of a sphere

(a) [8 pts] Compute all the non-vanishing components of the Riemann tensor R_{ijkl} $[(i, j, k, l) \in (\theta, \phi)]$ for the surface of a 2-sphere.

Solution: Though we could easily compute them from our formulas for a diagonal metric, it's convenient to use the mathematica notebook. The answer is conveniently summarized by

$$R_{\theta\phi\theta\phi} = a^2 \sin^2\theta \ . \tag{50}$$

Riemann symmetries provide all the other components.

(b) [8 pts] Consider the parallel transport of a tangent vector $\vec{A} = A^\theta\, \vec{e}_\theta + A^\phi\, \vec{e}_\phi$ on the sphere around an infinitesimal parallelogram of sides $\vec{e}_\theta\, d\theta$ and $\vec{e}_\phi\, d\phi$. Using the results of part (a), show that to first order in $d\Omega \equiv$

$\sin\theta\, d\theta\, d\phi$, the length of $\vec{A}$ is unchanged, but its direction rotates through an angle equal to $d\Omega$.

Solution: By the definition of the curvature tensor, to lowest order in the area of the parallelogram,

$$\tilde{A}^i = \left(\delta^i{}_j + R^i{}_{jkl} u^k v^l\right) A^j, \tag{51}$$

where u and v are the vectors which define the parallelogram.

The matrix

$$T^i{}_j \equiv \delta^i{}_j + R^i{}_{jkl} u^k v^l \tag{52}$$

is computed in the attached Mathematica notebook. The result is

$$T^i{}_j = \begin{pmatrix} 1 & d\theta d\phi \sin^2\theta \\ -d\theta d\phi & 1 \end{pmatrix}. \tag{53}$$

This transformation will preserve lengths (to first order) if

$$g_{ij} T^i{}_k T^j{}_l = g_{kl} \tag{54}$$

to lowest order in $d\theta d\phi$. We explicitly demonstrate this in the attached Mathematica notebook. Any transformation which preserves dot products (i.e. the metric) is, by definition, a rotation.

Explicitly acting on the vector, we find

$$A^\theta_T = A^\theta + \sin^2\theta\, d\theta\, d\phi\, A^\phi\ , \tag{55}$$

$$A^\phi_T = A^\phi - d\theta\, d\phi\, A^\theta\ , \tag{56}$$

where the subscript T means the vector after transport.

We can compute the angle of rotation using

$$\cos\psi \equiv \frac{A\cdot(TA)}{|A|\,|TA|} = \frac{A^i g_{ij} T^j{}_k A^k}{\sqrt{A^i g_{ij} A^j}\sqrt{T^i{}_k A^k g_{ij} T^j{}_l A^l}}. \tag{57}$$

To lowest order (explicitly computed in the Mathematica worksheet), we have

$$\cos\psi = 1 - \frac{\psi^2}{2} = 1 - \frac{1}{2}\left(\sin^2\theta d\theta^2 d\phi^2\right), \tag{58}$$

from which we identify $\psi = \sin\theta d\theta d\phi$.

(c) [8 pts] Show that, if $\vec{A}$ is parallel transported around the boundary of any simply connected solid angle Ω, its direction rotates through an angle Ω. ("Simply connected" is a topological term meaning that the boundary of the region could be shrunk to a point; it tells us that there are no holes in the manifold or other pathologies. See
`http://mathworld.wolfram.com/SimpleConnected.html`
for illustrations.) Compare with the result of Problem 3.

Solution: Full credit will be awarded just for the Stokes' Theorem explanation, but it's worth being aware of the fact that this trick only works in two dimensions.

First, let γ be the curve which bounds the region Ω. Let τ be a parameter for the curve γ (so we write $\gamma^i(\tau)$ for the coordinates of the point on the curve at parameter value τ, beginning from some arbitrary starting point). Then, the differential equation of parallel transport for $\vec{A}$ around the curve is

$$\frac{dA^i}{d\tau} = -\Gamma^i{}_{jk}(\gamma(\tau))\,\frac{d\gamma^j}{d\tau}A^k, \tag{59}$$

where we have made explicit that we are evaluating the connection coefficients along the curve γ. The following integral expression solves the differential equation, as can be verified using direct substitution:

$$A^i(\tau) = M^i{}_j(\tau)A^j(0), \tag{60}$$

where the transformation matrix M is given by

$$\begin{aligned} M^i{}_j(\tau) = \delta^i{}_j &+ \int_0^\tau ds\,\Gamma^i{}_{kj}(\gamma(s))\frac{d\gamma^k}{ds}(s) \\ &+ \int_0^\tau ds \int_0^s ds'\,\frac{d\gamma^m}{ds'}(s')\frac{d\gamma^k}{ds}(s)\Gamma^i{}_{kl}(\gamma(s))\Gamma^l{}_{mj}(\gamma(s')) + \ldots. \end{aligned} \tag{61}$$

M is composed of integrals of *path-ordered* products of Γ—the Γs with smaller path-length arguments appear to the right of those with larger. M is commonly written in "shorthand" as

$$M^i{}_j(\tau) = P\exp\left[\int_0^\tau ds\frac{d\gamma^m}{ds}(s)\Gamma^i{}_{mj}(s)\right], \tag{62}$$

where the P in front of the exponential stands for "path-ordered." It means that, when multiple integrals over the entire range $[0,\tau]$ appear in the series expansion of this matrix exponential, we remember to write the matrix product of the integrands in order of decreasing parameter argument. (The $1/n!$ which appears in the series expansion of the exponential — note that it *does not* appear in the explicit integral solution — accounts for the $n!$ different ways to re-arrange the products in decreasing parameter order.)

The total change in A upon parallel transport around the loop is, then,

$$A^i\left(\tau_f\right) - A^i(0) = \left[P\exp\left(\int_0^{\tau_f} ds\frac{d\gamma^m}{ds}(s)\Gamma^i{}_{mj}(s)\right) - \delta^i{}_j\right]A^i(0), \tag{63}$$

where τ_f is the smallest value of the curve parameter where $\gamma^i\left(\tau_f\right) = \gamma^i(0)$. We have reduced the problem of computing the change in A to computing the integral around the curve of $\Gamma \cdot d\gamma$ which appears in the exponential.

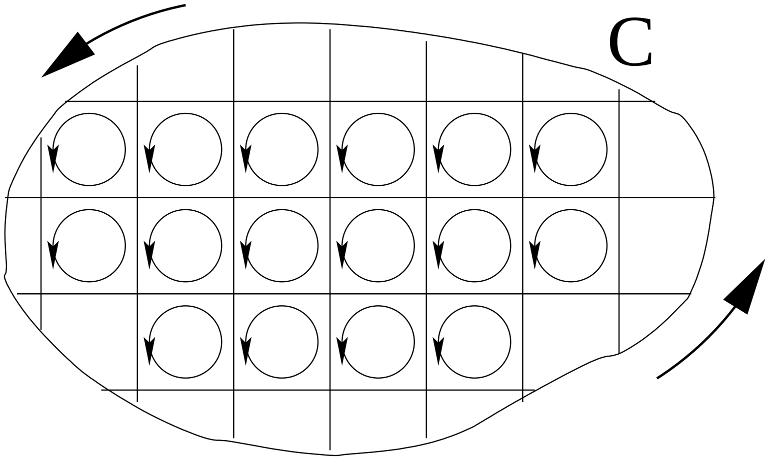


Figure 1: A prototypical integration contour and its decomposition into cells. The integral of a one-form evaluated along the contour will be equal to the integral of the one-form evaluated around each of the cells. The evaluations on interior boundaries of the cells cancel each other out.

If the quantity in the integral were a scalar, we could compute the integral using Stokes' Theorem. Consider the path outlined in figure 1. To evaluate the integral of a scalar along the path, we can instead evaluate the integral of the scalar around each of the cells. Interior boundaries of the cells are traversed twice, in opposite directions, so the total integral over the boundaries of the cells equals the integral along the curve.

Of course, we don't have a scalar quantity in our integral—we have a matrix. However, there is something special about the product of our matrix (the connection) around a single cell—this product is R, the curvature tensor, evaluated on the cell. *In two dimensions, the curvature tensor has only one free component, so, in a sense it is a scalar.* This is why this trick works in 2-D, but not in higher dimensions.

Using this heuristic argument, we can write equation (62) as

$$M^i{}_j\left(\tau_f\right) = \exp\left(\int_{\mathrm{int}(\gamma)} R^i{}_{jlm} dx^l \wedge dx^m\right). \tag{64}$$

But, we have already shown that, at an arbitrary point in the manifold, R evaluated on a cell generates an infinitesimal rotation by an angle equal to the solid angle of the cell. Therefore, the result of the integral is the sum of all these infinitesimal rotations—a finite rotation by an angle Ω.

5. [10 pts] Killing vectors and curvature

Prove that the relations

$$\begin{aligned}\nabla_\nu \nabla_\mu \xi^\alpha &= R^\alpha{}_{\mu\nu\beta}\xi^\beta && (65)\\ \Box\xi^\alpha &\equiv \nabla^\mu\nabla_\mu\xi^\alpha = -R^\alpha{}_\beta\xi^\beta && (66)\end{aligned}$$

are satisfied by any Killing vector ξ^α. This result is not too difficult to derive using the commutator $[\nabla_\alpha, \nabla_\beta] = \nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha$ and the identities

$$\begin{aligned}[\nabla_\alpha, \nabla_\beta]\, V^\mu &= R^\mu{}_{\nu\alpha\beta}V^\nu && (67)\\ [\nabla_\alpha, \nabla_\beta]\, V_\mu &= -R^\nu{}_{\mu\alpha\beta}V_\nu && (68)\end{aligned}$$

(If you are reading Schutz, these identities are incorrectly given on p. 171 — the minus sign is missing in the second identity.)

Solution: We know that

$$\begin{aligned}\nabla_\nu\nabla_\mu\xi_\alpha &= \left([\nabla_\nu, \nabla_\mu] + \nabla_\mu\nabla_\nu\right)\xi_\alpha && (69)\\ &= -R^\beta{}_{\alpha\nu\mu}\xi_\beta + \nabla_\mu\nabla_\nu\xi_\alpha && (70)\\ &= -R^\beta{}_{\alpha\nu\mu}\xi_\beta - \nabla_\mu\nabla_\alpha\xi_\nu, && (71)\end{aligned}$$

where the second line uses the definition of the curvature tensor, and the third line uses Killing's equation. We have rotated indices on the derivative at the cost of introducing a term with the curvature tensor and some minus signs. Repeating three times gives

$$\nabla_\nu\nabla_\mu\xi_\alpha = -\left(R^\beta{}_{\alpha\nu\mu} - R^\beta{}_{\nu\mu\alpha} + R^\beta{}_{\mu\alpha\nu}\right)\xi_\beta - \nabla_\nu\nabla_\mu\xi_\alpha. \qquad (72)$$

Using the identity $R^\alpha{}_{[\beta\gamma\delta]} = 0$, this can be transformed into

$$\nabla_\nu\nabla_\mu\xi_\alpha = R^\beta{}_{\nu\mu\alpha}\xi_\beta. \qquad (73)$$

Exploiting the various index-interchange identities of R, with some lowering and raising of indices, we have

$$R_{\beta\nu\mu\alpha} = R_{\mu\alpha\beta\nu} = R_{\alpha\mu\nu\beta}, \qquad (74)$$

which yields

$$\nabla_\nu\nabla_\mu\xi^\alpha = R^\alpha{}_{\mu\nu\beta}\xi^\beta. \qquad (75)$$

Contracting on μ and ν gives

$$\Box\xi^\alpha = R^\alpha{}_\beta\xi^\beta, \qquad (76)$$

which is what we were asked to show.

6. Riemann tensor for 1+1 static spacetimes

(a) [8 pts] Compute all the non-vanishing components of the Riemann tensor for the spacetime with line element

$$ds^2 = -e^{2\phi(x)}dt^2 + e^{-2\psi(x)}dx^2\ . \qquad (77)$$

Solution: This is a job for Mathematica. See the attached worksheet for the computation. The single independent non-zero component of R_{ijkl} is

$$R = e^{2\phi}\left(\phi'\left(\phi' + \psi'\right) + \phi''\right). \tag{78}$$

(b) [8 pts] For the case $\phi = \psi = \frac{1}{2}\ln|g(x - x_0)|$ where g and x_0 are constants, show that the spacetime is flat and find a coordinate transformation to globally flat coordinates $(\bar{t}, \bar{x})$ such that $ds^2 = -d\bar{t}^2 + d\bar{x}^2$.

Solution: The spacetime is definitely flat—evaluating the expression in equation (78), we find

$$R = e^{2\phi}\left(\phi'\left(\phi' + \psi'\right) + \phi''\right) = 0. \tag{79}$$

To find a coordinate transformation which transforms the metric to η, we seek a $\bar{t}$ and $\bar{x}$ which satisfy

$$\begin{aligned}
-\left(\frac{\partial \bar{t}}{\partial t}\right)^2 + \left(\frac{\partial \bar{x}}{\partial t}\right)^2 &= -g\left(x - x_0\right) && (80)\\
-\left(\frac{\partial \bar{x}}{\partial x}\right)^2 + \left(\frac{\partial \bar{x}}{\partial x}\right)^2 &= \frac{1}{g\left(x - x_0\right)} && (81)\\
-\frac{\partial \bar{t}}{\partial t}\frac{\partial \bar{t}}{\partial x} + \frac{\partial \bar{x}}{\partial t}\frac{\partial \bar{x}}{\partial x} &= 0 && (82)
\end{aligned}$$

so that

$$\eta_{\rho\sigma}\frac{\partial \bar{x}^\rho}{\partial x^\mu}\frac{\partial \bar{x}^\sigma}{\partial x^\nu} = g_{\mu\nu}. \tag{83}$$

A suitable transformation (guess and check works well here) is

$$\begin{aligned}
\bar{t}(t, x) &= \sqrt{\frac{4}{g}\left(x - x_0\right)}\cosh\left(\frac{gt}{2}\right) && (84)\\
\bar{x}(t, x) &= \sqrt{\frac{4}{g}\left(x - x_0\right)}\sinh\left(\frac{gt}{2}\right). && (85)
\end{aligned}$$