# 8.962 Problem Set 6 Solutions

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April 7, 2009

1. [15 pts] Constraint and evolution equations

   (Note: A long setup to a fairly short problem.)

   Maxwell's equations, written in terms of electric and magnetic fields and using cgs units with the speed of light $c = 1$, take the form

   $$\partial_i E^i = 4\pi\rho\ , \qquad \partial_i B^i = 0\ ; \tag{1}$$

   $$\partial_t E^i = \epsilon^{ij}{}_k \partial_j B^k - 4\pi J^i\ , \qquad \partial_t B^i = -\epsilon^{ij}{}_k \partial_j E^k\ . \tag{2}$$

   The first two equations — those relating the divergence of the fields — are called "constraint equations". This is because they involve no time derivatives — they are differential relations among the fields (and sources) at a single moment in time.

   The second two equations are called "evolution equations". Because they involve the time derivative of fields, they relate the fields and sources from one moment to the next. We can thus use the second pair of Maxwell equations to evolve "data" (which must satisfy the constraints) from some initial moment to an arbitrary late time.

   We will now prove that the Einstein field equations have a similar structure. Because the Einstein field equations are second order, we expect our "initial data" to consist of fields (the metric) and *first* time derivatives. Constraint equations should thus be those components of the Einstein equation which contain no more than a single time derivative; evolution equations are those components which contain two derivatives. (This is analogous to the kinematics of a particle: the "initial data" is the starting position and velocity; we use the acceleration as our "evolution equation" to find the particle's motion from then on.)

   Suppose we have chosen a timelike direction, so that $x^0 = t$; we do not specify the spatial coordinates $x^i$ other than to say that they are coordinates covering the $x^0 =$ constant hypersurface.

   Show that the Einstein tensor components $G^{00}$ and $G^{0i}$ contain no more than one time derivative. Thus, the equations $G^{00} = 8\pi G T^{00}$ and $G^{0i} = 8\pi G T^{0i}$ can be considered constraints which relate the metric, its first time derivative, and sources at a single moment of time; the equations $G^{ij} = 8\pi G T^{ij}$ are evolution equations.

(Hint: A brute force construction of the curvature tensor $G^{\alpha\beta}$ in terms of the metric and its derivatives will give you the correct solution to this problem. This is *not* a recommended procedure, though. A much quicker solution can be deduced by considering the Bianchi identity applied to the Einstein tensor.)

**Solution:** The contracted Bianchi identity states

$$\nabla_\mu G^{\mu\nu} = 0 \, . \tag{3}$$

We can expand this contraction to obtain

$$\nabla_0 G^{0\mu} + \nabla_i G^{i\mu} = 0 \, . \tag{4}$$

The first term will include a $\partial_0 G^{0\mu}$ term (in addition to some $\Gamma$-terms), while the second will have no derivatives with respect to time beyond those which already appear in $G$. In order that this identity be satisfied by all metrics, the largest number of $\partial_0$s appearing in each of the terms must agree. (I.e., if the first term contained a term like $\partial_0\partial_0\partial_0 g$, and the second had no compensating third time-derivative of $g$, then the Bianchi identity would take the form of a differential equation which imposes a constraint on $g$, not an identity which is satisfied for all possible $g$.) Therefore, the largest number of time-derivatives which appear in $G^{0\mu}$ must be one smaller than appear in $G^{i\mu}$.

We know that, in general, $G$ contains one derivative of the connection, $\Gamma$, which itself contains one derivative of $g$, so we must have some $\partial_0\partial_0 g$ terms in $G^{i\mu}$ and none in $G^{0\mu}$.

2. [12 pts] Action for a cosmological constant

Show that varying the action

$$S = \int d^4x \sqrt{-g}\,(R + a) \ , \tag{5}$$

(where $R$ is the Ricci scalar and $a$ is a constant) yields the Einstein equation with a cosmological constant. How does $a$ relate to the cosmological constant $\Lambda$?

**Solution:** We did the key pieces of algebra for this problem in class, showing that, up to boundary terms,

$$\delta \int d^4x \sqrt{-g}\, R = \int d^4x \sqrt{-g}\, G_{\mu\nu} \delta g^{\mu\nu} \tag{6}$$

when we vary the inverse metric by $g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu}$. The question, then, is what is

$$\delta \int d^4x \sqrt{-g}\, a. \tag{7}$$

From class notes (and your textbook), we know that

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \ . \tag{8}$$

Using this, we have

$$\delta\int d^4x\sqrt{-g}a = -\frac{a}{2}\int d^4x\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}, \tag{9}$$

which results in a modified Einstein equation:

$$G_{\mu\nu} - \frac{a}{2}g_{\mu\nu} = 0. \tag{10}$$

The Einstein equation with cosmological constant, $\Lambda$, reads

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \tag{11}$$

so we obtain $\Lambda = -a/2$.

3. Nordstrøm's gravity theory

A metric theory devised by G. Nordstrøm in 1913 (before general relativity was finalized) relates $g_{\mu\nu}$ to $T_{\mu\nu}$ by the equations

$$C_{\alpha\beta\gamma\delta} = 0\,, \qquad R = \kappa g_{\mu\nu}T^{\mu\nu} \tag{12}$$

where $C_{\alpha\beta\gamma\delta} = 0$ is the Weyl curvature tensor.

The vanishing of Weyl tells us that the metric is conformally flat; this follows from the fact that vanishing Riemann implies that spacetime is truly flat (not just conformally flat), and that the Weyl tensor is invariant under conformal transformations. Conformal flatness means that

$$g_{\mu\nu} = e^{2\phi}\eta_{\mu\nu} \ , \tag{13}$$

where $\phi = \phi(x^\mu)$ is a function of the spacetime coordinates. (To relate this to the notation used in lecture, $e^\phi = \Omega$; the exponential form is convenient for the calculations here.)

(a) [5 pts] Show that in the limits $\phi^2 \ll 1$ and $|\partial_t\phi| \ll |\partial_i\phi|$, the geodesic equation for a test body moving slowly ($u^i \ll 1$) in this spacetime reproduces the kinematics of Newtonian gravity. We'll call this the "Newtonian limit" from now on.

**Solution:** For a body which moves slowly, the geodesic equation reduces to

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{00}\frac{dt}{d\lambda}\frac{dt}{d\lambda} = 0, \tag{14}$$

where $\lambda$ is an affine parameter for the trajectory $x^\mu(\lambda)$, and $\Gamma$ is the Christoffel symbol. Since this is a diagonal metric, we can compute $\Gamma$

using formulas from Carroll (or using our various Mathematica tools). The result is

$$\Gamma^0{}_{00} = \mathcal{O}\left(\phi^2\right) \qquad \Gamma^i{}_{00} = \partial_i \phi \; . \tag{15}$$

The equation of motion for the spatial components of the trajectory becomes

$$\frac{d^2 x^i}{d\lambda^2} = -\partial_i \phi \left(\frac{dt}{d\lambda}\right)^2 , \tag{16}$$

or

$$\frac{d^2 x^i}{dt^2} = -\partial_i \phi \; . \tag{17}$$

This is exactly the Newtonian equation of motion.

(b) [10 pts] Show that in the Newtonian limit the Ricci scalar $R$ is just a second order differential operator acting on $\phi$. Compute that operator.

**Solution:** With a bit more grinding, we find

$$R = -6\nabla^2 \phi \; , \tag{18}$$

where $\nabla^2$ is the "usual" Euclidean geometry Laplace operator.

(c) [8 pts] Show that Nordstrøm's field equation reduces in the Newtonian limit to the gravitational field equations, and determine the value of $\kappa$.

**Solution:** For a dust (i.e. in the absence of pressure), assuming that $\rho \ll 1$, we have

$$g_{\mu\nu} T^{\mu\nu} \approx -\rho, \tag{19}$$

so the Nordstrøm equation becomes

$$-6\nabla^2 \phi = -\kappa \rho \; . \tag{20}$$

This gives us the correct Newtonian limit,

$$\nabla^2 \phi = 4\pi G \rho \; , \tag{21}$$

if we choose $\kappa = 24\pi G$.

(d) [7 pts] Is this theory consistent with the Pound-Rebka gravitational redshift experiment? (This is the experiment which established that light in fact redshifts as the equivalence principle predicts.)

**Solution:** The metric here is independent of time, which means that for any 4-momentum, the downstairs timelike component $p_t$ is conserved. This in turn tells us that

$$p_t \equiv g_{\mu\nu} \xi^\mu p^\nu = -e^{2\phi} p^0, \tag{22}$$

We can relate $p^0$ to the energy measured by an observer stationary in the given coordinate frame using $E_{\text{meas}} = -\vec{p} \cdot \vec{u}$,

$$E_{\text{meas}} = -g_{\mu\nu} u^\mu p^\nu = -e^{\phi} p^0 = e^{-\phi} E, \tag{23}$$

where $u^\mu \doteq (1/\sqrt{-g_{00}}, 0, 0, 0) = (e^{-\phi}, 0, 0, 0)$ is the four-velocity of the coordinate-stationary observer.

We see that $E_{\text{meas}}$ decreases as $\phi$ increases along a trajectory. In fact, for $\phi = gz$, light emitted at $z = 0$ with $E_{\text{meas}} = E$ will have

$$E_{\text{meas}}\big|_{z=h} = e^{-gh} E \approx (1 - gh)\, E, \tag{24}$$

which is exactly the redshift observed in the Pound-Rebka experiment.

(e) [7 pts] Show that there is no deflection of light by the sun in this theory of gravity.

**Solution:** Assume that light is instantaneously moving in the $x$ direction. Then $u^\mu = (1, 1, 0, 0)$ is its instantaneous four-velocity. The geodesic equation becomes (at that point)

$$\frac{du^\mu}{d\lambda} + \Gamma^\mu{}_{00} + \Gamma^\mu{}_{11} + 2\Gamma^\mu{}_{01} = 0. \tag{25}$$

Based on the Christoffel symbol derived in part a), we see that this implies

$$\begin{aligned}
\frac{du^0}{d\lambda} &= -2\partial_x\phi && (26)\\
\frac{du^1}{d\lambda} &= -2\partial_x\phi && (27)\\
\frac{du^2}{d\lambda} &= 0 && (28)\\
\frac{du^3}{d\lambda} &= 0, && (29)
\end{aligned}$$

so light which is instantaneously moving in the $x$ direction will continue moving in the $x$ direction forever. So, for any $\phi$ (including the $\phi$ which describes the sun), there is no deflection of light.

Note that we could have avoided the explicit calculation by noting that a conformal tranformation doesn't change the null geodesics of a spacetime. This means that light moves in the spacetime of the problem exactly as it moves in flat spacetime, and therefore does not bend around the sun (or any other object).

4. Weighing a relativistic body

An object of mass $m$ is at rest on a bathroom scale in a weak, uniform, static gravitational field. That is, the object has fixed spatial coordinates $(x, y, z)$ and the spacetime metric has the standard weak-field form $g_{\mu\nu} = \eta_{\mu\nu} - 2\phi\,\text{diag}(1, 1, 1, 1)$, with $\phi$ the normal Newtonian potential. We take $\phi^2 \ll 1$, $\partial_z\phi = \text{constant} = g$, and $\partial_\mu\phi = 0$ for $\mu \neq z$. Neglect terms of order $\phi^2$ and $\phi g$ in what follows. **NOTE: In the problem as assigned, I incorrectly gave the condition on the derivative as $\partial_z\phi = -g$. Aside from perhaps some confusion in interpretation (i.e., with**

**my original wording, bathroom scales go on the ceiling!), this does not affect the problem.**

In this problem, we will see that if one wants to interpret gravity as a force rather than as the effect of spacetime curvature, then it must be a velocity-dependent force. This is not a fundamental insight; the main purpose of the problem is to practice relating the metric to measurable quantities in curved spacetime.

(a) [5 pts] What force does the bathroom scale apply on the body? Compute both the components and the scalar magnitude of the 4-force. The principle to apply here is that the body does *not* follow a geodesic: The equation of motion for the body is

$$m\frac{D^2x^\mu}{d\tau^2} = mu^\beta\nabla_\beta u^\alpha = F^\mu \ . \tag{30}$$

This relation may be taken to *define* the 4-force $F^\mu$.

**Solution:** Up to an arbitrary (and irrelevant) constant, we have $\phi = gz$. We can use this to compute the Christoffel symbol (see the attached Mathematica worksheet).

The four-velocity of the body is

$$u^\mu \doteq \left(\frac{1}{\sqrt{-g_{00}}}, 0, 0, 0\right) \approx (1-\phi, 0, 0, 0). \tag{31}$$

All derivatives vanish in the geodesic equation, and we are left with (dropping terms of order $g^2$)

$$\frac{F^\mu}{m} \approx \Gamma^\mu{}_{00} \doteq (0, 0, 0, g). \tag{32}$$

A force of $g$ in the positive $z$-direction is exactly what we would expect. (Bear in mind that this is the force *on the body*, pushing the body from the geodesic it "wants" to follow.) The magnitude is $g$—because $F$ is already first-order in $g$, we can use the zero-order metric to compute the magnitude.

(b) [5 pts] Now suppose that the object moves with constant, relativistic coordinate 3-velocity $v = dx/dt = (dx/d\tau)(dt/d\tau)^{-1}$ in the $x$-direction:

$$V^x = vV^t \ ; \qquad V^y = V^z = 0 \ . \tag{33}$$

What is $V^t$? (Don't just use a special relativity formula!) While the mass is on the bathroom scale, what force (components and magnitude) does the scale apply to the mass?

**Solution:** The four-velocity must be normalized, so

$$g_{00}\left(V^t\right)^2 + g_{11}\left(V^x\right)^2 = \left(g_{00} + v^2 g_{11}\right)\left(V^t\right)^2 = -1, \tag{34}$$

which implies

$$V^t = \left[1 - v^2 + 2\phi\left(1 + v^2\right)\right]^{-1/2} \ . \tag{35}$$

Note, since the motion is relativistic we can't treat $v$ as small.

Since the mass moves with constant four-velocity, again all derivative terms vanish from the geodesic equation, and we have

$$\frac{F^\mu}{m} = \Gamma^\mu{}_{00}\left(V^t\right)^2 + \Gamma^\mu{}_{11}\left(V^x\right)^2 + 2\Gamma^\mu{}_{10}V^tV^x$$
$$\doteq \left(0, 0, 0, \frac{\left(1+v^2\right)g}{1-v^2}\right) . \tag{36}$$

Now the force's magnitude is $\left(1+v^2\right)g/\left(1-v^2\right)$.

(c) [5 pts] Now transform coordinates by applying a naive Lorentz transformation: $\bar{t} = \gamma(t - vx)$, $\bar{x} = \gamma(x - vt)$, $\bar{y} = y$, $\bar{z} = z$. Evaluate the components of the metric in the new coordinate system, $g_{\bar{\mu}\bar{\nu}}$. To first order in $\phi$, what are the force components in this new coordinate basis?

**Solution:** The new metric components are easily calculated using Mathematica (see the worksheet). The new force for a coordinate-stationary body is given by

$$\frac{F^{\hat{\mu}}}{m} = \Gamma^{\hat{\mu}}{}_{\hat{0}\hat{0}}, \tag{37}$$

which is explicitly calculated in the Mathematica worksheet. The result is

$$\frac{F^{\hat{\mu}}}{m} \doteq \left(0, 0, 0, \frac{\left(1+v^2\right)g}{1-v^2}\right) , \tag{38}$$

exactly as for the body which moves in the old coordinate system.

5. Converting between geometrized and "normal" units

Especially as we discuss astrophysical applications, we will find it useful to work in "geometrized units", in which the gravitational constant and the speed of light are both set to unity. When this is done, mass, length and time are measured in the same units.

We convert among different units by multiplying by powers of $G$ and $c$; since such a factor is just 1 in geometrized units, we can include as many such factors as is necessary. For example, to express the solar mass as a time, we write $M_\odot^{\rm geom} = GM_\odot/c^3$. Using

$$M_\odot = 1.99 \times 10^{33}\,\mathrm{gm}\ , \tag{39}$$
$$G = 6.67 \times 10^{-8}\,\mathrm{cm^3 gm^{-1} sec^{-2}}\ , \tag{40}$$
$$c = 3.00 \times 10^{10}\,\mathrm{cm/sec}\ , \tag{41}$$

we find $M_\odot^{\rm geom} = 4.93 \times 10^{-6}$ seconds.

Do the following conversions:

(a) [3 pts] Mass of the earth ($M_\oplus = 5.98 \times 10^{27}$ gm) in centimeters.

**Solution:** We have (Google is useful here, believe it or not)

$$M_\oplus^{\rm geom} = \frac{GM_\oplus}{c^2} = 0.44\,{\rm cm}. \tag{42}$$

(b) [3 pts] Characteristic mean density of neutron stars ($\bar\rho = 10^{15}$ gm/cm$^3$) in inverse square centimeters.

**Solution:**

$$\bar\rho^{\rm geom} = \frac{\bar\rho G}{c^2} = 7.42 \times 10^{-14}\,{\rm cm}^{-2} \tag{43}$$

(c) [3 pts] Characteristic mean pressure at core of a neutron star ($\bar P = 10^{34}$ gm sec$^{-2}$ cm$^{-1}$) in inverse square centimeters.

**Solution:**

$$\bar P^{\rm geom} = \frac{\bar P G}{c^4} = 8.26 \times 10^{-16}\,{\rm cm}^{-2} \tag{44}$$

(d) [3 pts] Acceleration of gravity at the surface of the earth ($g = 9.8$ m/s$^2$) in inverse seconds and inverse years.

**Solution:**

$$g^{\rm geom} = \frac{g}{c} = 3.2 \times 10^{-8}\,{\rm s}^{-1} = 1.03\,{\rm yr}^{-1} \tag{45}$$

(e) [3 pts] The typical (isotropic) luminosity of a gamma ray burst: $L = 10^{53}$ erg/sec. (Reminder: 1 erg = 1 gm cm$^2$/sec$^2$.)

**Solution:**

$$L^{\rm geom} = \frac{LG}{c^5} = 2.76 \times 10^{-7}. \tag{46}$$

(f) [3 pts] Planck's constant ($\hbar = 1.05 \times 10^{-27}$ erg sec) in square centimeters.

The square root of this quantity is called the "Planck length", and is denoted $l_p$. Since it involves the constants which set quantum effects ($\hbar$), gravitational effects ($G$), and relativistic effects ($c$), it is thought that quantum gravitational effects (i.e., the "quantization of spacetime", whatever that actually means) must be important at lengthscales $L \sim l_p$.

**Solution:**

$$l_p^2 = \frac{\hbar G}{c^3} = 2.6 \times 10^{-66}\,{\rm cm}^2 \tag{47}$$

(g) [3 pts] Convert $l_p$ to a mass (gm); this is known as the "Planck mass". Convert that to an energy; express this energy in electron volts. Comment on the likelihood of observing Planck mass scale effects at a particle collider.

**Solution:**

$$m_p = \frac{c^2 l_p}{G} = 2.17 \times 10^{-5}\,\text{gm}. \tag{48}$$

In energy:

$$E_p = m_p c^2 = 1.22 \times 10^{28}\,\text{eV}. \tag{49}$$

Given that the largest accelerators on Earth operate at the TeV (= $10^{12}$ eV) scale, this isn't going to be reached any time soon.