8.962 Problem Set 7 Solutions

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1. Gravitomagnetism

   In lecture and working in Lorentz gauge, we examined the linearized Einstein field equations for a static source,

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \quad \rightarrow \quad \nabla^2 \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \ , \tag{1}$$

   where ∇^2 is the ordinary Euclidean 3-space Laplacian operator. For a static, non-relativistic source, the only non-zero stress-energy component is (to sufficient accuracy for our purposes)

$$T_{00} = \rho \ . \tag{2}$$

   Using this, we found

$$\bar{h}_{00} = -4\Phi \rightarrow h_{\mu\nu} = -2\Phi \, \mathrm{diag}(1,1,1,1) \ , \tag{3}$$

   where $\Phi = -GM/r$ is the Newtonian gravitational potential.

   We will now modify this slightly by imagining that the source rotates, and thus is characterized by a spin angular momentum with spatial components S^i as well as a mass M.

   (a) [7 pts] Consider the source to be spherically symmetric, with uniform density ρ and radius R. Take it to be rotating rigidly about the $x^3 \equiv z$ axis with constant angular velocity Ω. Working in a Lorentz frame that is at rest with respect to the center of mass of the source, work out all components of the stress energy tensor $T_{\mu\nu}$ to first order in Ω. (Assume ρ, R, and Ω are constant.) Indicate which components would change if you included terms to second order in Ω, but don't calculate those second order corrections. (You may neglect pressure terms throughout your calculation.)

   Solution: Neglecting pressure, the stress energy tensor is

$$T_{\mu\nu} = \rho u_\mu u_\nu \tag{4}$$

   where ρ is the rest density of the source's fluid. To first order in Ω, we thus have

$$\begin{aligned} T_{00} &= \rho \ , & (5) \\ T_{0i} &= \rho v_i = \rho v^i \ , & (6) \\ T_{ij} &= 0 \ . & (7) \end{aligned}$$

Working in a Cartesian basis, $v^x = y\Omega$, $v^y = -x\Omega$, $v^z = 0$.

The components T_{00} and T_{ij} would be modified if $\mathcal{O}(\Omega^2)$ terms were kept.

(b) Solve for the Cartesian off-diagonal components h_{0x}, h_{0y}, h_{0z}. (Note that $h_{0i} = \bar{h}_{0i}$ since trace reversal has no effect on off-diagonal components.)

This is a moderately challenging calculation. The following tips should help:

- Recall that the formal solution to the Poisson-type equation for h_{0i} is

$$h_{0i}(\mathbf{x}) = 4G \int \frac{T_{0i}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' \tag{8}$$

  where $\mathbf{x}$ is the "field point", the location of the point at which h_{0i} is to be evaluated, and $\mathbf{x}'$ is the "source point", a coordinate within the source over which the integral is taken. [Boldface quantities denote 3-vectors: $\mathbf{x} \doteq (x, y, z)$.]

- The following expansion for the factor $1/|\mathbf{x} - \mathbf{x}'|$ is very useful:

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{r} + \frac{x^j x^{j'}}{r^3} + \ldots \tag{9}$$

  You may assume this identity in your solution. Note also that a sum over j is implied here; we are allowed to be sloppy about the placement of indices since the spatial metric is δ_{ij} to leading order. [This identity is more often seen as an expansion in spherical harmonics; see, for example, J. D. Jackson, Sec. 3.6 (2nd edition). This form in terms of Cartesian coordinates is equivalent.]

- After you have set up your integral, convert the primed integration variable to spherical coordinates to do the integration:

$$x^{1'} = x' \quad \rightarrow \quad r' \sin\theta' \cos\phi' \tag{10}$$

$$x^{2'} = y' \quad \rightarrow \quad r' \sin\theta' \sin\phi' \tag{11}$$

$$x^{3'} = z' \quad \rightarrow \quad r' \cos\theta' \tag{12}$$

Your final metric components should be proportional to $\rho R^5/r^3$.

Solution: Taking advantage of the tips, we know that

$$h_{0i} = 4G\rho \left[\frac{1}{r} \int v^i(\mathbf{x}')\, d^3x' + \frac{x^j}{r^3} \int x^{j'} v^i(\mathbf{x}')\, d^3x' \right] . \tag{13}$$

You should be able to quickly convince yourself that the first term vanishes since $v^i(\mathbf{x}')$ is an odd function of $\mathbf{x}'$. Let us expand the second integral, focusing on h_{0x}. Inserting $v^x = y'\Omega$ and expanding all the other terms, we find

$$\begin{aligned} h_{0x} &= 4G\Omega\rho\left[\frac{x}{r^3}\int x'y'\,dx'dy'dz' + \frac{y}{r^3}\int (y')^2\,dx'dy'dz' \right. \\ &\qquad \left. + \frac{z}{r^3}\int z'y'\,dx'dy'dz'\right] . \end{aligned} \tag{14}$$

The first and third integrals vanish due to the oddness of their integrands. We rewrite the second integral in spherical coordinates, putting $y' = r'\sin\theta'\sin\phi'$, $dx'dy'dz' = (r')^2 dr'\sin\theta' d\theta' d\phi'$:

$$h_{0x} = \frac{4G\Omega\rho y}{r^3}\int_0^R dr'\int_0^\pi d\theta'\int_0^{2\pi} d\phi'(r')^4(\sin\theta')^3(\sin\phi')^2 \tag{15}$$

$$= \frac{16\pi G\Omega\rho y R^5}{15 r^3} . \tag{16}$$

By a similar calculation, we find

$$h_{0y} = -\frac{16\pi G\Omega\rho x R^5}{15 r^3} , \tag{17}$$

$$h_{0z} = 0 . \tag{18}$$

(c) Using the identity $S^i = I\Omega^i$ where I is moment of inertia and Ω^i is the ith component of the angular velocity vector, rewrite your answer in terms of the angular momentum S^i.

Solution: First, note that the total mass of the source is $4\pi\rho R^3/3$, so our answers may be rewritten

$$h_{0x} = \frac{4G\Omega y M R^2}{5r^3} , \tag{19}$$

$$h_{0y} = -\frac{4G\Omega x M R^2}{5r^3} , \tag{20}$$

$$h_{0z} = 0 . \tag{21}$$

Next, we use the fact that the moment of inertia of a rigidly rotating, uniform density sphere is $I = 2MR^2/5$:

$$h_{0x} = \frac{2GyS^z}{r^3} , \tag{22}$$

$$h_{0y} = -\frac{2GxS^z}{r^3} , \tag{23}$$

$$h_{0z} = 0 . \tag{24}$$

We have used the fact that the angular velocity is along the z axis to put $S^z = I\Omega$.

Although we derived this result for a special situation (uniform density, spherical body, rigid rotation), the result we obtain in terms of S^i is completely general; see, for example, MTW Sec. 19.1.

(d) Converting to spherical coordinates, find h_{0r}, $h_{0\theta}$, $h_{0\phi}$.

Solution: Our transformation equation is

$$h_{0\bar{k}} = \frac{\partial x^i}{\partial x^{\bar{k}}} h_{0i} \tag{25}$$

where $x^{\bar{k}} \doteq (r, \theta, \phi)$, $x^i \doteq (x, y, z)$. Taking the derivatives, we see that this equation is nicely written in the matrix form

$$\begin{pmatrix} h_{0r} \\ h_{0\theta} \\ h_{0\phi} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{pmatrix} \cdot \begin{pmatrix} h_{0x} \\ h_{0y} \\ h_{0z} \end{pmatrix} . \tag{26}$$

Inserting $h_{0x} = 2GS^z \sin\theta\sin\phi/r^2$, $h_{0y} = -2GS^z \sin\theta\cos\phi/r^2$, $h_{0z} = 0$, we find

$$h_{0r} = 0 \tag{27}$$

$$h_{0\theta} = 0 \tag{28}$$

$$h_{0\phi} = -\frac{2GS^z \sin^2\theta}{r} . \tag{29}$$

2. Comparison of linearized GR and Maxwell's theory

Consider the line element

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)(dx^2 + dy^2 + dz^2) + 2\beta^i \, dx^i \, dt \; ; \tag{30}$$

in other words, the usual weak field line element on the diagonal with $h_{0i} = \beta^i$. (To first order in deviations from flat spacetime, we also have $\beta^i = \delta^{ij}\beta_j = \beta_i$, and also $h^{0i} = -\beta^i$.)

(a) [10 pts] Show that the geodesic equation for a particle moving in this spacetime gives the following equation of motion to first order in the particle's velocity $\mathbf{v}$:

$$m\frac{d^2\mathbf{x}}{dt^2} = m\mathbf{g} + m(\mathbf{v} \times \mathbf{H}) . \tag{31}$$

Here, $\mathbf{x}$ is a 3-vector representing the position of the particle, and

$$\mathbf{g} = -\boldsymbol{\nabla}\Phi , \tag{32}$$

$$\mathbf{H} = \boldsymbol{\nabla} \times \boldsymbol{\beta} , \tag{33}$$

where $\boldsymbol{\nabla}$ represents the ordinary gradient operator in Euclidean 3-space.

Solution: The trick here is to gather terms at the right order in our linearized theory and at the right order in our non-relativistic limit. Let us introduce an order-counting parameter into our spacetime: We put

$$ds^2 = -(1+2\epsilon\Phi)dt^2 + (1-2\epsilon\Phi)(dx^2+dy^2+dz^2) + 2\epsilon\beta^i\,dx^i\,dt\ . \quad (34)$$

We then want to keep quantities describing our spacetime to $O(\epsilon)$, and quantities describing our test particle to $O(v)$, where v is the test particle's typical speed.

Let us first write out the geodesic equation. We have

$$m\frac{d^2x^i}{d\tau^2} + m\Gamma^i{}_{00}\frac{dt}{d\tau}\frac{dt}{d\tau} + 2m\Gamma^i{}_{0j}\frac{dx^j}{d\tau}\frac{dt}{d\tau} + O(v^2) = 0\ , \quad (35)$$

or

$$m\frac{d^2x^i}{dt^2} + m\Gamma^i{}_{00} + 2m\Gamma^i{}_{0j}\frac{dx^j}{dt} = 0\ . \quad (36)$$

Using GRTool.nb, I find

$$\begin{aligned}\Gamma^i{}_{00} &= \boldsymbol{\nabla}_i\Phi + O(\epsilon^2) &(37)\\ &= -\mathbf{g}_i\ , &(38)\end{aligned}$$

and

$$\begin{aligned}\Gamma^x{}_{0x} &= 0\ , &(39)\\ \Gamma^x{}_{0y} &= \frac{1}{2}\left(\partial_y\beta^x - \partial_x\beta^y\right)\ , &(40)\\ \Gamma^x{}_{0z} &= \frac{1}{2}\left(\partial_z\beta^x - \partial_x\beta^z\right)\ . &(41)\end{aligned}$$

Defining

$$\mathbf{H} = \boldsymbol{\nabla}\times\boldsymbol{\beta}\ , \quad (42)$$

the results $\Gamma^x{}_{0j}$ can be written

$$\Gamma^x{}_{0j} = -\frac{1}{2}\epsilon_{xjk}\mathbf{H}_k\ . \quad (43)$$

Working out the other components, we have more generally

$$\Gamma^i{}_{0j} = -\frac{1}{2}\epsilon_{ijk}\mathbf{H}_k\ . \quad (44)$$

Now, put all this together to build our equation of motion:

$$m\frac{d^2x^i}{dt^2} = m\mathbf{g}_i + m\epsilon_{ijk}\frac{dx^j}{dt}\mathbf{H}_k\ , \quad (45)$$

or

$$m\frac{d^2\mathbf{x}}{dt^2} = m\mathbf{g} + m\mathbf{v}\times\mathbf{H}\ . \tag{46}$$

(b) [10 pts] Show that for stationary sources (i.e., no component of the stress energy tensor shows time variation) the Einstein field equations may be written

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{g} &= -4\pi G\rho\ , &(47)\\
\boldsymbol{\nabla}\times\mathbf{H} &= -16\pi G\boldsymbol{J} &(48)\\
\boldsymbol{\nabla}\cdot\mathbf{H} &= 0\ , &(49)\\
\boldsymbol{\nabla}\times\mathbf{g} &= 0\ . &(50)
\end{aligned}$$

The current $\mathbf{J} = \rho\mathbf{v}$, where $\mathbf{v}$ is the velocity of fluid flow in the source. (Note that the second two equations follow from the definitions of $\mathbf{g}$ and $\mathbf{H}$, so the only labor is in working out the first two.)

Solution: Using GRTool.nb (or a similar tool), work out the components of the Einstein tensor. To $O(\epsilon)$, we find

$$G_{00} = 2\nabla^2\Phi = -2\boldsymbol{\nabla}\cdot\mathbf{g} \tag{51}$$

Using $T_{00} = 8\pi\rho$, we find

$$\boldsymbol{\nabla}\cdot\mathbf{g} = -4\pi G\rho\ . \tag{52}$$

We also find

$$G_{0i} = -\frac{1}{2}\nabla^2\beta^i\ . \tag{53}$$

Actually, not quite: You only find this *if* you choose $\partial_i\beta^i = 0$. However, recall that the linearized Einstein equations were derived in Lorentz gauge, for which we chose $\partial_i\bar{h}^{ij} = 0$. For $j = 0$, this amounts to the requirement that $2\partial_t\Phi - \partial_i\beta^i = 0$. Since Φ is time-independent, this condition is true.

This lets us write this as

$$G_{0i} = -\frac{1}{2}\left(\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\beta}\right)_i\ . \tag{54}$$

Using $T_{0i} = 8\pi\rho v_i$, we find

$$\boldsymbol{\nabla}\times\mathbf{H} = -16\pi G\rho\mathbf{v}\ . \tag{55}$$

The other components equaling zero follows simply from vector identities applied to the definitions of $\mathbf{H}$ and $\mathbf{g}$.

Note: In the course of doing this problem, some people were finding that they got equations with β^i on both the right-hand side and the left-hand side of the equation. Terms such as this came about by unconsciously including terms that are non-linear in the metric in your analysis. For example, using the solution we have sketched here, β is of order $(G\rho)$. An equation which includes β on the right-hand side (acting as a source) could then be solved iteratively. The solution we have found here would be the leading order solution; it would act as a source for a correction of order $(G\rho)^2$.

Such an iterative scheme is in fact used quite a lot, and is known as the "post-Newtonian" approach to general relativity. Careful use of order-counting parameters (such as the ϵ used here) helps to keep things at consistent order in the calculation.

(c) [5 pts] These equations clearly bear a strong resemblance to Maxwell's equations in the limit $\partial_t \mathbf{E} = \partial_t \mathbf{B} = 0$; the main differences are the reversed sign in both equations, and the extra factor of 4 (compared to Maxwell) in the curl equation. Can you give a simple explanation for these differences?

Solution: The sign difference occurs because two positive masses *attract* each other, instead of repelling as two positive charges would. In E&M, the opposite signs on these terms allow one to form a wave equation for $\mathbf{E}$ and $\mathbf{B}$; here the dynamical part of the Einstein equation is the ij-components.

The factor of 4 is related to the fact that our "potential" is a 2-index tensor rather than a 1-index vector; this in turn is fundamentally due to the fact that gravity is spin 2 while electromagnetism is spin-1.

3. [13 pts] Carroll: Chapter 7, Problem 1.

In this problem, Carroll asks us to vary a certain Lagrangian to construct the linearized Einstein tensor [Carroll Eq. (7.8)]. If you vary this Lagrangian in the "obvious" way, you will probably find that you get *almost* the correct Einstein tensor — you should get Eq. (7.8), but with the first two terms replaced with 2 times the first term. In other words, you don't get the symmetrization on μ and ν that we should get.

What is going on here? The issue is that the Lagrangian doesn't "know", *a priori*, that the tensor $h_{\mu\nu}$ is symmetric. This has a strong impact on the second term of the Lagrangian — it should be symmetric with respect to exchange of the indices ρ and σ, but isn't unless you somehow build in the knowledge we have of this symmetry.

There are two simple ways to address this:

a. Rewrite the Lagrangian to force this symmetrization:

$$(\partial_\mu h^{\rho\sigma})(\partial_\rho h^\mu{}_\sigma) \to \frac{1}{2}\left[(\partial_\mu h^{\rho\sigma})(\partial_\rho h^\mu{}_\sigma) + (\partial_\mu h^{\rho\sigma})(\partial_\sigma h^\mu{}_\rho)\right] \; ; \qquad (56)$$

b. Make sure that, in our variation, this symmetry is enforced. The way

I did this was to note that I should have

$$\frac{\delta(\partial_\mu h_{\rho\sigma})}{\delta(\partial_\gamma h_{\alpha\beta})} = \frac{\delta(\partial_\mu h_{\rho\sigma})}{\delta(\partial_\gamma h_{\beta\alpha})} \tag{57}$$

$$= \frac{1}{2}\left[\frac{\delta(\partial_\mu h_{\rho\sigma})}{\delta(\partial_\gamma h_{\alpha\beta})} + \frac{\delta(\partial_\mu h_{\rho\sigma})}{\delta(\partial_\gamma h_{\beta\alpha})}\right] \tag{58}$$

$$= \frac{1}{2}\left[\delta^\gamma{}_\mu \delta^\alpha{}_\rho \delta^\beta{}_\sigma + \delta^\gamma{}_\mu \delta^\alpha{}_\sigma \delta^\beta{}_\rho\right] . \tag{59}$$

It shouldn't be too difficult to convince yourself that these methods are in fact equivalent.

Solution: Given the Lagrangian

$$2L \equiv (\partial_\mu h^{\mu\nu})(\partial_\nu h) - (\partial_\mu h^{\rho\sigma})(\partial_\rho h^\mu{}_\sigma) + \frac{1}{2}\eta^{\mu\nu}(\partial_\mu h^{\rho\sigma})(\partial_\nu h_{\rho\sigma}) - \frac{1}{2}\eta^{\mu\nu}(\partial_\mu h)(\partial_\nu h), \tag{60}$$

we can just grind through the variations.

A note about the hint: when varying Lagrangians, you need to be very careful about what your configuration space is. You are only allowed to make variations in your fields (or trajectories, or whatever) which *remain in configuration space*. For example, in one-dimensional mechanics problems you generally have a configuration space which consists of smooth paths between two points, so your variation must also be smooth to preserve this property. In our case, the field is a metric, so our configuration space is the space of possible metrics. This is the space of 4×4, *symmetric* matrices, so our variation must be a 4×4 symmetric matrix, too.

Grinding away: we will need to discover the variations in $\partial_\mu h^{\nu\rho}$, $\partial_\mu h^{\mu\nu}$, and $\partial_\mu h$ when $h^{\mu\nu} \to h^{\mu\nu} + \delta h^{\mu\nu}$, with $\delta h^{\mu\nu} = \delta h^{(\mu\nu)}$. Since all these expressions are linear in $h^{\mu\nu}$, this is pretty straightforward:

$$\delta(\partial_\mu h^{\nu\rho}) = \partial_\mu \delta h^{\nu\rho} \tag{61}$$

$$\delta(\partial_\mu h^{\mu\nu}) = \partial_\mu \delta h^{\mu\nu} \tag{62}$$

$$\delta(\partial_\mu h) = \eta_{\nu\rho}\partial_\mu \delta h^{\nu\rho}. \tag{63}$$

Plugging in, and exploiting that the last two terms in the Lagrangian are symmetric in their h-dependence, we obtain

$$2\delta L = (\partial_\mu \delta h^{\mu\nu})(\partial_\nu h) + (\partial_\mu h^{\mu\nu})\eta_{\rho\sigma}(\partial_\nu \delta h^{\rho\sigma}) - (\partial_\mu \delta h^{\rho\sigma})(\partial_\rho h^\mu{}_\sigma) - (\partial_\mu h^{\rho\sigma})\eta_{\nu\sigma}(\partial_\rho \delta h^{\mu\nu}) + \eta^{\mu\nu}(\partial_\mu h^{\rho\sigma})(\partial_\nu \delta h_{\rho\sigma}) - \eta^{\mu\nu}\eta_{\rho\sigma}(\partial_\mu \delta h^{\rho\sigma})(\partial_\nu h). \tag{64}$$

Now we will apply the generalized divergence theorem:

$$\int_V d^n x \partial_\mu T^{\mu\nu\cdots} = \int_{\partial V} d^{n-1} x n_\mu T^{\mu\nu\cdots}, \tag{65}$$

where n_μ is the normal to the boundary of the region of integration, V. On the boundary of our spacetime region, we require that the variation $\delta h^{\mu\nu} = 0$. So, for example, we have

$$\begin{aligned}
\int_V d^4x\ (\partial_\mu \delta h^{\mu\nu})(\partial_\nu h) &= \int_V d^4x\, \partial_\mu (\delta h^{\mu\nu} \partial_\nu h) && (66)\\
&\quad - \int_V d^4x\ (\delta h^{\mu\nu} \partial_\mu \partial_\nu h) && (67)\\
&= \int_{\partial V} d^3x\, n_\mu (\delta h^{\mu\nu} \partial_\nu h) && (68)\\
&\quad - \int_V d^4x\ (\delta h^{\mu\nu} \partial_\mu \partial_\nu h) && (69)\\
&= -\int_V d^4x\ (\delta h^{\mu\nu} \partial_\mu \partial_\nu h) && (70)
\end{aligned}$$

when the action integral is applied. Transforming every term with a derivative this way, we obtain

$$\begin{aligned}
2\delta L \sim -\big[\partial_\mu \partial_\nu h + \eta_{\mu\nu} \partial_\alpha \partial_\beta h^{\alpha\beta} - 2\partial_\alpha \partial_\mu h^\alpha{}_\nu \\
+\eta^{\alpha\beta} \partial_\alpha \partial_\beta h_{\mu\nu} - \eta^{\alpha\beta} \eta_{\mu\nu} \partial_\alpha \partial_\beta h\big]\, \delta h^{\mu\nu}, \quad (71)
\end{aligned}$$

where the notation $\sim$ means "is the same variation under the action integral". We want to eliminate the $\delta h^{\mu\nu}$ using the fundamental theorem of variational calculus, but before we do, we must ensure that the object which is contracted with $\delta h^{\mu\nu}$ is symmetric. If it isn't, we won't be adhering to our rule about only generating variations which respect the configuration space once we remove the contraction, though the above expression, where the anti-symmetric part of the expression in brackets vanishes, is entirely correct.

In equation (71), the only anti-symmetric term is the one with the coefficient of 2. Re-writing that term symmetrically, we find that the action is stationary only when

$$\partial_\mu \partial_\alpha h^\alpha{}_\nu + \partial_\nu \partial_\alpha h^\alpha{}_\mu - \partial_\mu \partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu} \partial_\alpha \partial_\beta h^{\alpha\beta} - \eta_{\mu\nu} \Box h = 0, \quad (72)$$

but this is exactly Carroll's expression for $G_{\mu\nu}$ (equation 7.8).

4. [12 pts] Carroll: Chapter 7, Problem 4.

Note: It's easy to get tripped up in this problem by thinking of x^μ as the components of a vector. Doing so will make the problem far more complicated than it needs to be. The idea instead is to regard the index μ as labeling members of a set of scalar functions: Each coordinate is just some scalar function, and our goal is to show that "box" applied to the scalar x^μ is equivalent to Lorentz gauge in linearized theory. In practice, this means you should ignore the fact that x^μ has an index when you expand the box operator. (As a notational mnemonic, you might want to replace x^μ with some scalar function f until you have expanded all the covariant derivatives into partials and connection coefficients.)

Solution: We are asked to show that the Lorenz gauge condition,

$$\partial_\mu \bar{h}^\mu_\nu = 0 \tag{73}$$

is equivalent to the harmonic gauge condition

$$\Box x^\mu = g^{\rho\nu} \nabla_\rho \nabla_\nu x^\mu = 0 \tag{74}$$

in the linear limit. We will assume the harmonic gauge condition, and show that it implies the Lorenz gauge condition via a reversible argument.

The coordinate functions have particularly simple derivative:

$$\nabla_\nu x^\mu = \partial_\nu x^\mu = \delta_\nu{}^\mu, \tag{75}$$

where we are only viewing the lower index as a "tensorized" index; the upper index is just a set enumerator for $\{0, 1, 2, 3\}$. This has zero derivative, so

$$\nabla_\rho \delta_\nu{}^\mu = -\Gamma^\alpha{}_{\rho\nu} \delta_\alpha{}^\mu = -\Gamma^\mu{}_{\rho\nu} = 0. \tag{76}$$

Thus, harmonic gauge reduces to

$$\Box x^\mu = 0 = -g^{\rho\nu} \Gamma^\mu{}_{\rho\nu}. \tag{77}$$

In the linear limit, this becomes

$$\Box x^\mu = 0 = \eta^{\rho\nu} \eta^{\mu\kappa} \left[\partial_\rho h_{\kappa\nu} + \partial_\nu h_{\kappa\rho} - \partial_\kappa h_{\rho\nu} \right]. \tag{78}$$

Contracting on ρ and ν, we obtain

$$\Box x^\mu = 0 = \eta^{\mu\kappa} \left[2\partial_\nu h^\nu{}_\kappa - \partial_\kappa h \right]. \tag{79}$$

Relabeling a few indices and contracting the η on the first term, we have

$$\Box x^\mu = 0 = 2\partial_\nu h^{\nu\mu} - \partial_\nu \eta^{\nu\mu} h = 2\partial_\nu \bar{h}^{\nu\mu}, \tag{80}$$

which demonstrates that harmonic gauge is equivalent in the linear limit to Lorenz gauge.