8.962 Problem Set 8 Solutions

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1. [15 pts] In lecture, we derived the following formula for the leading gravitational radiation generated by a source:

$$h_{ij}^{\mathrm{TT}} = \frac{2G\ddot{I}_{kl}}{r}\left(P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\right) . \tag{1}$$

Here, overdot denotes d/dt, $I_{kl} = \int dV\, \rho\, x^k x^l$, we are ignoring the distinction between upstairs and downstairs indices since we are in a nearly flat region and can work in nearly inertial coordinates, and the combination of projection tensors guarantees that the resulting tensor is transverse and traceless.

Show that the same result is obtained if one uses the traceless "quadrupole moment tensor" $\mathcal{I}_{kl} = I_{kl} - \frac{1}{3}\delta_{kl} I$ instead of I_{jk}, where $I = \delta_{kl} I_{kl}$.

Comment 1: This is less trivial than it may seem since there are really two different trace operations defined here: A trace with respect to the spatial metric $\eta_{ij} = \delta_{ij}$, and a trace with respect to the metric of the subspace orthogonal to the propagation direction, P_{ij}.

Comment 2: In general, a radiative l-pole has $2l+1$ separate components — a scalar has one component, a dipole has three, a quadrupole has five. The symmetric spatial tensor I_{ij} has 6 components — there must be "extra" information in that tensor unrelated to radiation. This exercise proves that this extra information is bound up in the trace. For this reason, you will often see the formula for h_{ij}^{TT} written in terms of $\mathcal{I}_{ij}$ rather than I_{ij}.

Solution: We want to show that

$$\ddot{I}_{kl}\left(P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\right) = \ddot{\mathcal{I}}_{kl}\left(P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\right), \tag{2}$$

i.e. that it makes no difference whether we use I or $\mathcal{I}$ in the gravitational radiation formula. Examining the right-hand-side, we have

$$\ddot{\mathcal{I}}_{kl}\left(P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\right) = \left(I_{kl} - \frac{1}{3}\delta_{kl} I\right)\left(P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\right). \tag{3}$$

If we can show that

$$\delta_{kl}\left(P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\right) = 0, \tag{4}$$

then we would be justified in replacing I with $\mathcal{I}$. Multiplying out the δs, we have

$$\delta_{kl}\left(P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\right) = P_{il}P_{jl} - \frac{1}{2}P_{ij}P = P_{ij} - \frac{P}{2}P_{ij}, \tag{5}$$

where we have used the fact that $\mathbf{P}^2 = \mathbf{P}$, and we define $P \equiv \delta_{ij}P_{ij}$. Hopefully, $P = 2$! Let's check:

$$P = \delta_{ij}\left(g_{ij} - n_i n_j\right) = 3 - 1 = 2. \tag{6}$$

Because $P = 2$, the trace term in $\mathcal{I}_{ij} = I_{ij} - 1/3\delta_{ij}I$ does not contribute to the radiation formula, and we have

$$h_{ij}^{\mathrm{TT}} = \frac{2G\ddot{\mathcal{I}}_{kl}}{r}\left(P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\right). \tag{7}$$

2. Binary system

Consider a binary consisting of two masses m_1 and m_2 in a circular orbit about one another; the separation of the bodies is R. Consider the orbit to be adequately described using Newtonian gravity; in this problem, we will use this description to compute the leading effects due to gravitational-wave emission. (We use Newtonian gravity to give the short-timescale orbital kinematics; we use the quadrupole gravitational waves to compute how this orbit evolves on long timescales.)

[*Hint:* Don't forget that orbits in a problem of this type are most easily described using the "reduced system": a body of mass $\mu = m_1m_2/(m_1 + m_2)$ in circular orbit of *radius* R around a body of mass $M = m_1 + m_2$. If you need a refresher, this result is derived in all junior-level mechanics textbooks; see, for example, Goldstein Sec. 3.1.]

(a) [8 pts] Compute the gravitational-wave tensor h_{ij}^{TT} as measured by an observer looking down the angular momentum axis of the system (i.e., the z-axis if you define the orbital plane as the $x - y$ plane).

Solution: [Note: the solution to a special case of this problem can be found in Carroll, Chapter 7, Sections 5 and 6. When comparing, be sure to note that Carroll does the equal-mass case, so $M_{\mathrm{us}} = 2M_{\mathrm{Carroll}}$ and $\mu = M_{\mathrm{Carroll}}/2$. Also, Carroll uses orbital "radius," where we use orbital separation; $R_{\mathrm{us}} = 2R_{\mathrm{Carroll}}$.] To solve this problem, we need to compute the stress-energy tensor component T^{00} for this system. The orbit of the bodies in time is given by

$$\vec{r}_1 = \frac{R\mu}{m_1}\left(\cos\left(\Omega t\right), \sin\left(\Omega t\right), 0\right) \tag{8}$$

$$\vec{r}_2 = -\frac{R\mu}{m_2}\left(\cos\left(\Omega t\right), \sin\left(\Omega t\right), 0\right), \tag{9}$$

where $\Omega = \sqrt{GM/R^3}$ is the Keplerian orbital frequency for the system. The masses generate stress-energy at their positions, so we have

$$T^{00} = m_1\delta^3\left(\vec{x} - \vec{r}_1(t)\right) + m_2\delta^3\left(\vec{x} - \vec{r}_2(t)\right), \tag{10}$$

and the quadrupole tensor is

$$I^{ij} = \int d^3x T^{00} x^i x^j = m_1 r_1^i r_1^j + m_2 r_2^i r_2^j. \tag{11}$$

In matrix form, this is

$$\mathbf{I} = R^2\mu \begin{pmatrix} \cos^2(\Omega t) & \cos(\Omega t)\sin(\Omega t) & 0 \\ \cos(\Omega t)\sin(\Omega t) & \sin^2(\Omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{12}$$

It will be convenient to re-write this without the powers and products; exploiting simple trigonometric identities, we have

$$\mathbf{I} = \frac{1}{2}R^2\mu \begin{pmatrix} 1+\cos(2\Omega t) & \sin(2\Omega t) & 0 \\ \sin(2\Omega t) & 1-\cos(2\Omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{13}$$

The quadrupole formula then gives for the trace-reversed metric (which in this case is just the metric, since it's traceless):

$$\begin{aligned} \bar{\mathbf{h}}^{TT}(t,\mathbf{x}) = \mathbf{h}^{TT}(t,\mathbf{x}) &= \frac{2G}{r}\ddot{\mathbf{I}}(t_r) \\ &= \frac{4GR^2\Omega^2\mu}{r} \begin{pmatrix} -\cos\left(2\Omega t_r\right) & -\sin\left(2\Omega t_r\right) & 0 \\ -\sin\left(2\Omega t_r\right) & \cos\left(2\Omega t_r\right) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{14}$$

where $t_r = t - |\mathbf{x}|$.

(b) [7 pts] Compute the rate at which energy is carried away from the system by gravitational waves.

Due to this loss of energy, the radius of the orbit will gradually shrink, and the frequency of the binary will "chirp" to higher frequencies as time passes.

Solution: The power radiated by gravitational waves is

$$P = -\frac{G}{5}\left\langle \dddot{\mathcal{I}}_{ij}\dddot{\mathcal{I}}^{ij}\right\rangle, \tag{15}$$

where you should note that we have used the traceless quadrupole tensor instead of I. $\mathcal{I}$ is defined by

$$\mathcal{I}_{ij} = I_{ij} - \frac{1}{3} I \delta_{ij}. \tag{16}$$

Since $I \equiv \delta_{ij} I^{ij}$ is time-independent in this case, derivatives of $\mathcal{I}$ and I will agree, so we have

$$P = -\frac{G}{5} \left\langle \dddot{I}_{ij} \dddot{I}^{ij} \right\rangle = -\frac{32 G R^4 \mu^2 \Omega^6}{5} = -\frac{32 G^4 M^3 \mu^2}{5 R^5} \tag{17}$$

(c) [7 pts] By asserting *global* conservation of energy in the following form,

$$\frac{d}{dt} \left(E_{\text{kinetic}} + E_{\text{potential}} + E_{\text{GW}} \right) = 0 , \tag{18}$$

derive an equation for dR/dt, the rate at which the orbital separation shrinks.

[*Hint:* Don't forget that for circular, Newtonian orbits, there is a simple relationship expressing $E_{\text{kinetic}} + E_{\text{potential}}$ as a function of R, as well as a simple result for the orbital frequency as a function of R (Kepler's 3rd law).]

Solution: We have (from the virial theorem) that

$$E_{\text{kinetic}} + E_{\text{potential}} = -\frac{G M \mu}{2R} \tag{19}$$

so

$$\frac{d}{dt} \left(E_{\text{kinetic}} + E_{\text{potential}} \right) = \frac{G M \mu}{2R^2} \frac{dR}{dt}. \tag{20}$$

In the last part, we worked out that

$$\frac{d}{dt} E_{\text{GW}} = -P = \frac{32 G R^4 \mu^2 \Omega^6}{5} = \frac{32 G^4 M^3 \mu^2}{5 R^5}. \tag{21}$$

Global conservation of energy requires, then, that

$$\frac{dr}{dt} = -\frac{64 G^3 M^2 \mu}{5 R^3}. \tag{22}$$

(d) [6 pts] Derive the rate of change of the orbital angular frequency, Ω, caused by gravitational-wave emission. You should find that the masses only appear in the combination $\mathcal{M} \equiv \mu^{3/5} M^{2/5}$, perhaps raised to some power. This combination of masses is known as the "chirp mass", since it sets the rate at which the frequency "chirps".

Solution: We have

$$\Omega = \sqrt{GM} R^{-3/2}, \tag{23}$$

so

$$\frac{d\Omega}{dt} = -\frac{3}{2}\sqrt{\frac{GM}{R^5}}\frac{dR}{dt} = \frac{96\mu}{5}\sqrt{\frac{G^7M^5}{R^{11}}} = \frac{96}{5}\left(G\mathcal{M}\Omega^{11/5}\right)^{5/3} \tag{24}$$

(e) [4 pts] Integrate the $d\Omega/dt$ you obtained in part (d) to obtain $\Omega(t)$, the time evolution of the binary's orbital frequency. Let $T_{\rm coal}$ the "coalescence time" be the time at which the frequency (formally) goes to infinity. (In reality, the various approximations we have introduced into this calculation break down before we reach this time; $T_{\rm coal}$ is nonetheless a fairly accurate stand-in for the time at which the members of a binary merge due to gravitational-wave emission.) Your answer should be a power law in $T_{\rm coal} - t$.

Comment: Modulo a correction to account for the effect of eccentricity, the $\Omega(t)$ you just derived is what is observed in the orbit of relativistic binary pulsars. Measuring this decay and showing that it agreed with general relativity's prediction is in part what led to the Nobel Prize award for Russell Hulse and Joe Taylor in 1993.

Solution: Integrating the differential equation derived for Ω above, we have

$$\Omega(t) = \frac{15^{3/8}}{2^{9/8}}\frac{1}{\left(C - 96G^{5/3}\mathcal{M}^{5/3}t\right)^{3/8}}, \tag{25}$$

where C is an arbitrary constant of integration. Expressing C in terms of $T_{\rm coal}$, we have

$$\Omega(t) = \frac{5^{3/8}}{8}\frac{1}{G^{5/8}\mathcal{M}^{5/8}}\frac{1}{(T_{\rm coal} - t)^{3/8}} = \left[\frac{5}{256}\frac{1}{G^{5/3}\mathcal{M}^{5/3}}\frac{1}{T_{\rm coal} - t}\right]^{3/8} \tag{26}$$

3. Wave equation for the Riemann tensor in linearized theory

As we have emphasized from time to time, there is a nice analogy between the metric of GR and the electromagnetic potential, and likewise between curvature tensors and the electromagnetic field. This suggests that it should be possible to build a wave equation for the curvature tensor.

As background to this problem and the next one, recall that the Einstein field equations can be written in the trace-reversed form

$$R^{\mu}{}_{\alpha\mu\beta} = R_{\alpha\beta} = 8\pi G\bar{T}_{\alpha\beta}$$

where $\bar{T}_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T^{\gamma}{}_{\gamma}$.

To keep things simple, we begin with linearized theory, working in nearly inertial coordinates: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $||h_{\mu\nu}|| \ll 1$.

(a) [5 pts] To linear order in h, the Bianchi identity can be written

$$\partial_\alpha R_{\beta\gamma\mu\nu} + \partial_\beta R_{\gamma\alpha\mu\nu} + \partial_\gamma R_{\alpha\beta\mu\nu} = 0\ . \tag{27}$$

Using this equation, show that the divergence of the Riemann tensor is related to the gradient of the trace-reversed stress energy tensor:

$$\partial_\alpha R^\alpha{}_{\beta\gamma\delta} = 8\pi G \left[\text{source involving gradient of } \bar{T}_{\mu\nu}.\right] \tag{28}$$

Solution: At first, it seems tempting to contract the Bianchi identity above on α and β (which would turn the first term into a divergence), but that turns out not to be very useful: The third term in the Bianchi identity is identically zero on contraction (thanks to symmetry-antisymmetry), and the first term terms simply cancel each other. You get $0 = 0$, a true statement but not exactly useful.

A bit of reflection will convince you that we must contract on one index which appears as a derivative in equation (27) and one index which does not. We choose (arbitrarily) to contract on α and μ. First, let's apply the symmetries of Riemann to generate the terms we want upon contraction:

$$\partial_\alpha R_{\beta\gamma\mu\nu} + \partial_\beta R_{\gamma\alpha\mu\nu} + \partial_\gamma R_{\alpha\beta\mu\nu} = \partial_\alpha R_{\mu\nu\beta\gamma} - \partial_\beta R_{\alpha\gamma\mu\nu} + \partial_\gamma R_{\alpha\beta\mu\nu} = 0. \tag{29}$$

Now contraction on α and μ gives

$$\partial_\alpha R^\alpha{}_{\nu\beta\gamma} = \partial_\beta R_{\gamma\nu} - \partial_\gamma R_{\beta\nu} = 8\pi G \left(\partial_\beta \bar{T}_{\gamma\nu} - \partial_\gamma \bar{T}_{\beta\nu}\right). \tag{30}$$

(b) [6 pts] Now use the Bianchi identity and the solution to part (a) to develop a wave equation for the Riemann tensor of the form

$$\Box R_{\alpha\beta\mu\nu} = 8\pi G \left[\text{source involving double gradients of } \bar{T}_{\mu\nu}.\right] \tag{31}$$

Solve this equation (formally) using the radiative Green's function introduced in lecture.

Solution: This time we can just apply another derivative, $\partial_{\alpha'}$, to equation (27) and then contract on α and α' to obtain (applying some symmetries to put the result in more pleasing form)

$$\Box R_{\beta\gamma\mu\nu} - \partial_\beta \partial_\alpha R^\alpha{}_{\gamma\mu\nu} + \partial_\gamma \partial_\alpha R^\alpha{}_{\beta\mu\nu} = 0. \tag{32}$$

We can apply the results of part (a) to re-write this in terms of gradients of $\bar{T}$. First, for convenience, define

$$\bar{T}_{\mu\nu\rho\sigma} = \partial_\mu \partial_\nu \bar{T}_{\rho\sigma}. \tag{33}$$

Then we have

$$\Box R_{\beta\gamma\mu\nu} = 8\pi G \left(\bar{T}_{\gamma\mu\nu\beta} - \bar{T}_{\gamma\nu\mu\beta} + \bar{T}_{\beta\nu\mu\gamma} - \bar{T}_{\beta\mu\nu\gamma}\right) \equiv 8\pi G S_{\beta\gamma\mu\nu}. \tag{34}$$

Then, using the Green's function introduced in lecture, we have

$$R_{\beta\gamma\mu\nu}(t, \mathbf{x}) = -2G \int d^3y \frac{S_{\beta\gamma\mu\nu}\left(t_r, \mathbf{y}\right)}{|\mathbf{x} - \mathbf{y}|}, \tag{35}$$

where $t_r = t - |\mathbf{x} - \mathbf{y}|$.

(c) [7 pts] Now specialize to a plane gravitational wave propagating in the z-direction through vacuum. The corresponding solution to the above wave equation takes the form $R_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu}(t - z)$. Using the Bianchi identity and the symmetries of Riemann, show that the only non-zero components of the Riemann tensor are of the form R_{i0j0} (plus components that are trivially related by symmetries; recall that indices i, j only refer to spatial indices). Note, this analysis will be greatly facilitated by expanding the Riemann tensor in plane waves: Put

$$R_{\alpha\beta\mu\nu} = C_{\alpha\beta\mu\nu} e^{ik_\sigma x^\sigma} \tag{36}$$

with

$$k^\sigma \doteq (\omega, 0, 0, \omega) \ . \tag{37}$$

This changes the Bianchi identity into a simple algebraic relation between components of Riemann and components k^σ.

Solution: [Note that this problem should have asked us to show that the components R_{i0j0} are the only independent components of R—all other components are either identically zero, or can be expressed in terms of R_{i0j0}.] Under the plane-wave Fourier decomposition, the Bianchi identity becomes an algebraic equation involving the wavenumber, k:

$$k_\alpha R_{\beta\gamma\mu\nu} + k_\beta R_{\gamma\alpha\mu\nu} + k_\gamma R_{\alpha\beta\mu\nu} = 0. \tag{38}$$

In our coordinate system, we have

$$k^\alpha \doteq (\omega, 0, 0, \omega) \, , \tag{39}$$

where ω is the frequency of the particular plane wave we're considering. Letting $\alpha = 0$ in the above identity, we have

$$R_{\beta\gamma\mu\nu} = -\frac{1}{\omega} \left(k_\beta R_{\gamma 0\mu\nu} - k_\gamma R_{\beta 0\mu\nu} \right) . \tag{40}$$

We can further expand each of the terms on the RHS of this equation by applying the identity again. If we apply it to, for example, $R_{\gamma 0\mu\nu}$, we get nothing new. However, if we apply it to $R_{\mu\nu\gamma 0}$ (which is equal to $R_{\gamma 0\mu\nu}$ by symmetries) and to the corresponding combination of indices in the second term, we obtain

$$R_{\beta\gamma\mu\nu} = \frac{1}{\omega^2} \left(k_\gamma k_\nu R_{\mu 0\beta 0} - k_\beta k_\nu R_{\mu 0\gamma 0} - k_\gamma k_\mu R_{\nu 0\beta 0} + k_\beta k_\mu R_{\nu 0\gamma 0} \right) . \tag{41}$$

But, because Riemann is anti-symmetric in consecutive pairs of indices, the R terms here do not contribute unless they are of the form R_{i0j0}. Thus, we have shown that every non-zero component of R can be expressed in terms of R_{i0j0}.

(d) [4 pts] Show that the only non-zero R_{i0j0} are $R_{x0x0}(t-z) = -R_{y0y0}(t-z)$ and $R_{x0y0}(t-z) = R_{y0x0}(t-z)$. The first non-zero components correspond to the + polarization discussed in lecture; the second corresponds to the $\times$ polarization.

Solution: Consider the 00-component of the Einstein equation in vacuum:

$$R_{00} = R^x{}_{0x0} + R^y{}_{0y0} + R^z{}_{0z0} = 0. \tag{42}$$

We can compute the third term in the sum using the identity in part (c). The result is that $R_{z0z0} = 0$, so the 00-component of the Einstein equation gives

$$R_{x0x0} = -R_{y0y0}. \tag{43}$$

The identity from (c) also implies that all R of the form $R_{z0\mu\nu} = 0$, so the only remaining independent component which can be non-zero is $R_{0x0y} = R_{0y0x}$. This component is un-constrained by the identity in part (c).

Thus, the only non-zero independent components of R for a wave propagating in the z-direction are $R_{0x0x} = -R_{0y0y}$ and $R_{0x0y} = R_{0y0x}$.

(e) [5 pts] Define fields $h_+(t-z)$ and $h_\times(t-z)$ in terms of these components of the Riemann tensor by

$$R_{x0x0} = -\frac{1}{2}\partial_t^2 h_+ \,, \qquad R_{x0y0} = -\frac{1}{2}\partial_t^2 h_\times \,. \tag{44}$$

Also recall the expression for the Riemann tensor in linearized theory:

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left(\partial_\alpha\partial_\nu h_{\beta\mu} + \partial_\beta\partial_\mu h_{\alpha\nu} - \partial_\alpha\partial_\mu h_{\beta\nu} - \partial_\beta\partial_\nu h_{\alpha\mu}\right) \tag{45}$$

Comparing these two forms, show that $h_+ = h^{\rm TT}_{xx} = -h^{\rm TT}_{yy}$, $h_\times = h^{\rm TT}_{xy} = h^{\rm TT}_{yx}$.

Solution: We just need to plug in the appropriate values for α, β, μ, ν, and we see that

$$R_{x0x0} = -\frac{1}{2}\partial_t^2 h_{11} + \text{terms with } h_{0\mu} \tag{46}$$

and

$$R_{x0y0} = -\frac{1}{2}\partial_t^2 h_{12} + \text{term with } h_{0\mu}. \tag{47}$$

Assuming that we have gauge-fixed all $h_{0\mu}$ terms to zero, we obtain the desired identities.

(f) [6 pts] Show that when one rotates the coordinate system about the waves' propagation direction (i.e., about the z-axis) by an angle θ [so that $x' + iy' = (x+iy)e^{-i\theta}$], the gravitational-wave fields h_+ and $h_\times$ transform such that

$$h'_+ + ih'_\times = (h_+ + ih_\times)e^{-i2\theta} \tag{48}$$

This equation is equivalent to the statement that the graviton is spin 2 — the quantity $h_+ + ih_\times$ has "spin-weight" 2.

Solution: Since we earlier showed that we could use the usual h_+ and $h_\times$ as a substitute for the degrees of freedom available in R, we will perform this analysis on the metric components only. Under a coordinate rotation with the Jacobian

$$J^i{}_j = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad (49)$$

the metric transforms by

$$h_{i'j'} = J^i{}_{i'} J^j{}_{j'} h_{ij}. \quad (50)$$

If we have

$$h_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad (51)$$

then we obtain

$$h_{i'j'} = \begin{pmatrix} h_+ \cos(2\theta) + h_\times \sin(2\theta) & h_\times \cos(2\theta) - h_+ \sin(2\theta) & 0 \\ h_\times \cos(2\theta) - h_+ \sin(2\theta) & -h_+ \cos(2\theta) - h_\times \sin(2\theta) & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad (52)$$

so we see that $h'_+ + ih'_\times = \exp(-2i\theta)\,(h_+ + ih_\times)$.

4. Nonlinear wave equation for the Riemann tensor

Begin now with the full Bianchi identity:

$$\nabla_\alpha R_{\beta\gamma\mu\nu} + \nabla_\beta R_{\gamma\alpha\mu\nu} + \nabla_\gamma R_{\alpha\beta\mu\nu} = 0 \; . \quad (53)$$

(a) [10 pts] Develop the fully covariant analog to your answer to part (a) of problem (3):

$$\nabla_\alpha R^\alpha{}_{\beta\gamma\delta} = 8\pi G \left[\text{source involving covariant gradient of } \bar{T}_{\mu\nu}.\right] \quad (54)$$

Solution: The solution from the last problem goes through without any changes beyond $\partial \mapsto \nabla$.

(b) [10 pts] Using the Bianchi identity and the solution to part (a), develop a nonlinear wave equation for the Riemann tensor of the form

$$\Box R^\alpha{}_{\beta\gamma\delta} = 8\pi G\,[\text{source}] \; . \quad (55)$$

In this case, the source should involve three kinds of terms: double covariant gradients of $\bar{T}_{\mu\nu}$, coupling of $\bar{T}_{\mu\nu}$ to the Riemann tensor, and coupling of Riemann to Riemann. Here, the operator $\Box = \nabla^\mu \nabla_\mu$.

Comment: This non-linear wave equation was first developed by Roger Penrose, and is sometimes called the Penrose wave equation. A variant of this equation based on an expansion around the Riemann tensor for a black hole spacetime was developed by Saul Teukolsky and has played an extremely important role in astrophysical relativity research.

Solution: Here we have to be a bit more careful, because covariant derivative operators do not commute. The Bianchi identity plus one contracted derivative gives

$$\Box R_{\beta\gamma\mu\nu} = \nabla_\alpha \nabla_\beta R^\alpha{}_{\gamma\mu\nu} - \nabla_\alpha \nabla_\gamma R^\alpha{}_{\beta\mu\nu}. \tag{56}$$

We would like to express this using terms like $\nabla_\alpha R^\alpha{}_{\beta\gamma\delta}$ because then we could apply the result from part (a). To do so, we need to consider the commutator of covariant derivatives acting on R (which we can compute by analogy with the defining identity for R on vectors):

$$\begin{aligned}\left[\nabla_\mu, \nabla_\nu\right] R^\alpha{}_{\beta\gamma\rho} = R^\alpha{}_{\sigma\mu\nu} R^\sigma{}_{\beta\gamma\rho} - R^\sigma{}_{\beta\mu\nu} R^\alpha{}_{\sigma\gamma\rho} \\ - R^\sigma{}_{\gamma\mu\nu} R^\alpha{}_{\beta\sigma\rho} - R^\sigma{}_{\rho\mu\nu} R^\alpha{}_{\beta\gamma\sigma}\end{aligned} \tag{57}$$

Using this, we can commute the derivatives on the RHS of our wave equation. Once we have them in the form of $\nabla_{\{\beta,\gamma\}} \nabla_\alpha R^\alpha{}_{\{\gamma,\beta\}\mu\nu}$, we can apply the results from the first part to obtain the result

$$\begin{aligned}&\Box R_{\beta\gamma\mu\nu} = \\ &\quad 8\pi G \left(\nabla_\beta \nabla_\mu \bar{T}_{\nu\gamma} - \nabla_\beta \nabla_\nu \bar{T}_{\mu\gamma} - \nabla_\gamma \nabla_\mu \bar{T}_{\nu\beta} + \nabla_\gamma \nabla_\nu \bar{T}_{\mu\beta}\right) \\ &\quad - R^\alpha{}_{\sigma\alpha\gamma} R^\sigma{}_{\beta\mu\nu} + R^\alpha{}_{\sigma\mu\nu} R^\sigma{}_{\beta\alpha\gamma} - R^\alpha{}_{\sigma\mu\nu} R^\sigma{}_{\gamma\alpha\beta} \\ &\quad + R^\alpha{}_{\sigma\alpha\beta} R^\sigma{}_{\gamma\mu\nu} - R^\alpha{}_{\gamma\sigma\nu} R^\sigma{}_{\mu\alpha\beta} + R^\alpha{}_{\beta\sigma\nu} R^\sigma{}_{\mu\alpha\gamma} \\ &\quad - R^\alpha{}_{\gamma\mu\sigma} R^\sigma{}_{\nu\alpha\beta} + R^\alpha{}_{\beta\mu\sigma} R^\sigma{}_{\nu\alpha\gamma}.\end{aligned} \tag{58}$$

All the contracted Riemanns here can be written as the stress-energy tensor:

$$\begin{aligned}&\Box R_{\beta\gamma\mu\nu} = \\ &\quad 8\pi G \left(\nabla_\beta \nabla_\mu \bar{T}_{\nu\gamma} - \nabla_\beta \nabla_\nu \bar{T}_{\mu\gamma} - \nabla_\gamma \nabla_\mu \bar{T}_{\nu\beta} + \nabla_\gamma \nabla_\nu \bar{T}_{\mu\beta}\right) \\ &\quad - 8\pi G \bar{T}_{\sigma\gamma} R^\sigma{}_{\beta\mu\nu} + R^\alpha{}_{\sigma\mu\nu} R^\sigma{}_{\beta\alpha\gamma} - R^\alpha{}_{\sigma\mu\nu} R^\sigma{}_{\gamma\alpha\beta} \\ &\quad + 8\pi G \bar{T}_{\sigma\beta} R^\sigma{}_{\gamma\mu\nu} - R^\alpha{}_{\gamma\sigma\nu} R^\sigma{}_{\mu\alpha\beta} + R^\alpha{}_{\beta\sigma\nu} R^\sigma{}_{\mu\alpha\gamma} \\ &\quad - R^\alpha{}_{\gamma\mu\sigma} R^\sigma{}_{\nu\alpha\beta} + R^\alpha{}_{\beta\mu\sigma} R^\sigma{}_{\nu\alpha\gamma}.\end{aligned} \tag{59}$$