

8.962 Pset 10 Solutions

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1. Isotropic representation of the Schwarzschild metric

(a) [15 pts] The Schwarzschild line element, written in Schwarzschild coordinates, takes the form

$$ds^2 = - \left(1 - \frac{2GM}{r} \right) dt^2 + \frac{dr^2}{(1 - 2GM/r)} + r^2 d\Omega^2 . \quad (1)$$

Show that changing to the radial coordinate \bar{r} , defined by

$$r = \bar{r} \left(1 + \frac{GM}{2\bar{r}} \right)^2 , \quad (2)$$

puts the Schwarzschild line element into the form

$$ds^2 = g_{tt}(\bar{r}) dt^2 + g(\bar{r})(d\bar{r}^2 + \bar{r}^2 d\Omega^2) . \quad (3)$$

This new coordinate system is called “isotropic coordinates”, since it emphasizes the fundamental local isotropy of the three spatial directions.

Compute the metric functions $g_{tt}(\bar{r})$ and $g(\bar{r})$.

Solution: Some useful quantities:

$$1 - \frac{2GM}{r} = \left(\frac{1 - \frac{GM}{2\bar{r}}}{1 + \frac{GM}{2\bar{r}}} \right)^2 , \quad (4)$$

and

$$dr = \left(1 - \frac{GM^2}{4\bar{r}^2} \right) d\bar{r} . \quad (5)$$

Substituting, we find

$$ds^2 = - \left(\frac{1 - \frac{GM}{2\bar{r}}}{1 + \frac{GM}{2\bar{r}}} \right)^2 dt^2 + \left(1 + \frac{GM}{2\bar{r}} \right)^4 (d\bar{r}^2 + \bar{r}^2 d\Omega^2) , \quad (6)$$

from which we can read off

$$g_{tt}(\bar{r}) = \left(\frac{1 - \frac{GM}{2\bar{r}}}{1 + \frac{GM}{2\bar{r}}} \right)^2 , \quad (7)$$

and

$$g(\bar{r}) = \left(1 + \frac{GM}{2\bar{r}}\right)^4. \quad (8)$$

(b) [5 pts] Take the limit $\bar{r} \gg GM$. Show that the line element then reduces to a form appropriate for describing the exterior of a spherical body in the linearized limit of general relativity.

Solution: The exterior of a spherical body in weak-field GR has the metric

$$ds^2 = -(1 - 2\Phi)dt^2 + (1 + 2\Phi)(dr^2 + r^2d\Omega^2). \quad (9)$$

Expanding the expression in equation (6) for the isotropic Schwarzschild metric, and keeping only the lowest power of $1/\bar{r}$, we obtain

$$ds^2 \approx -\left(1 - \frac{2GM}{\bar{r}}\right)dt^2 + \left(1 + \frac{2GM}{\bar{r}}\right)(d\bar{r}^2 + \bar{r}^2d\Omega^2), \quad (10)$$

which is of the correct form. (Note that equation (2) implies that $\bar{r} \rightarrow \infty$ as $r \rightarrow \infty$, so this is the limit infinitely far from the Schwarzschild mass.)

2. Numerical construction of neutron star models in GR

A moderately accurate approximation to the equation of state of the material which makes up a neutron star is given by the polytropic form

$$P = K\rho_0^\Gamma \quad (11)$$

where P is the pressure, ρ_0 is the *rest* matter density, and the constants are given by

$$\Gamma = 5/3, \quad (12)$$

$$K = \frac{3^{2/3}\pi^{4/3}}{5} \frac{\hbar^2}{m_n^{8/3}} \quad (13)$$

$$= 5.38 \times 10^9 \text{ (dyne/cm}^2\text{)(gm/cm}^3\text{)}^{-5/3}. \quad (14)$$

$$= 5.38 \times 10^9 \text{ gm}^{-2/3} \text{ cm}^4 \text{ sec}^{-2}. \quad (15)$$

In this problem, you will numerically integrate the TOV equations of stellar structure to build models of neutron stars in general relativity. Your goal will be to examine the total mass M_* and radius R_* as a function of central density $\rho_{0,c}$. As such, you *must* familiarize yourself with some system for numerically solving a system of ordinary differential equations. My personal experience has shown that the routine `NDSolve` built into the package `MATHEMATICA` is well suited to this problem. (An example of using `NDSolve` will be posted to the 8.962 website, to illustrate how it works.) More ambitious students with sufficient programming and numerical methods expertise may prefer to write their own code in `C` or `C++`, or whatever you are most comfortable with.

During your calculations, don't forget to take into account the difference between rest mass density ρ_0 and relativistic energy density ρ :

$$\rho = \rho_0 + \frac{P}{\Gamma - 1} = \rho_0 + \frac{K\rho_0^\Gamma}{\Gamma - 1}. \quad (16)$$

The equations you will integrate take the form

$$\frac{dm}{dr} = 4\pi\rho r^2 \quad (17)$$

$$\frac{dP}{dr} = -\frac{(\rho + P)(m + 4\pi r^3 P)}{r(r - 2m)}. \quad (18)$$

We have here written these equations in units in which both G and c are set equal to 1. The initial conditions you need to apply are

$$m(r = 0) = 0, \quad (19)$$

$$P(r = 0) = P_c = P(\rho_{0,c}) = K\rho_{0,c}^\Gamma. \quad (20)$$

One integrates these equations until the pressure drops to zero: $P(R_*) = 0$ defines the star's surface. (In fact, you will probably find in your numerical integration that MATHEMATICA or your code attempts to take an infinite number of infinitesimally small steps in the vicinity of R_* . The radius at which this numerical inaccuracy sets in will still define the surface of the star to very high accuracy.) The total mass of the star is then defined as $m(R_*) \equiv M_*$.

(a) [6 pts] Experience has shown that numerical calculations of this kind behave best when the units are chosen such that many key quantities are within an order of magnitude or so of unity. A good choice is to put $G = 1$ and $c = 1$; we then pick the units of all dimensionful quantities to be powers of kilometers. With this choice, the order of magnitude of the total mass will be roughly 1 kilometer, and the order of magnitude of the stellar radius will be roughly 10 kilometers.

Using this unit system, convert

(i) $\rho = 1 \text{ gm/cm}^3$ to km^{-2} .

(ii) $P = 1 \text{ dyne/cm}^2 = 1 \text{ gm cm}^{-1} \text{ sec}^{-2}$ to km^{-2} .

(iii) $K = 1 \text{ gm}^{-2/3} \text{ cm}^4 \text{ sec}^{-2}$ to $\text{km}^{4/3}$. Use this conversion factor to convert the K that appears in the polytropic equation of state to $\text{km}^{4/3}$.

Solution: We have

$$\frac{\rho G}{c^2} = 7.42 \times 10^{-19} \text{ km}^{-2} \quad (21)$$

$$\frac{PG}{c^4} = 8.26 \times 10^{-40} \text{ km}^{-2} \quad (22)$$

$$KG^{-2/3}c^{-2/3} = 1.36 \times 10^{-9} \text{ km}^{4/3}. \quad (23)$$

The last of these relations implies that

$$KG^{-2/3}c^{-2/3} = 7.32 \text{ km}^{4/3} \quad (24)$$

(b) [15 pts] Pick a central density $\rho_{0,c} = 10^{15} \text{ gm/cm}^3$. Inserting appropriate conversion factors such that all quantities are in units of km (or powers of km), integrate the TOV equations to compute the radius R_* and mass M_* of the star that has this central density.

Solution: Using $\rho_{0,c} = 10^{15} \text{ g/cm}^3$ gives $\rho_{0,c}G/c^2 = 7.42 \times 10^{-4} \text{ km}^{-2}$. The remainder of this solution is done in the attached Mathematica worksheet.

We find that

$$R_* = 12.6885 \text{ km} \quad (25)$$

$$M_* = 1.00509 \text{ km} = 0.68063 M_\odot. \quad (26)$$

(c) [10 pts] If a photon is emitted radially with energy E_{em} from the surface of this star, what is the energy E_{ob} with which this photon is observed by distant ($r \rightarrow \infty$) observers? Using these energies, compute the surface redshift

$$z_{\text{surf}} = \frac{E_{\text{ob}} - E_{\text{em}}}{E_{\text{ob}}}. \quad (27)$$

Solution: The energy of a photon with four-momentum p^μ measured by an observer with four-velocity u^μ is given by

$$E = -u \cdot p = -p_\mu u^\mu. \quad (28)$$

The photon is emitted with an energy E_{em} ; this means that an observer fixed to the star's surface (i.e. an observer at $r = R_*$, with four-velocity $u \doteq (1/\sqrt{-g_{tt}}, \vec{0})$) measures

$$E_{\text{em}} = -p_\mu u^\mu = -\frac{p_0}{\sqrt{-g_{tt}}} \quad (29)$$

or

$$p_0 = -E_{\text{em}}\sqrt{-g_{tt}} = -E_{\text{em}}\sqrt{1 - \frac{2GM_*}{R_*}}. \quad (30)$$

Since this spacetime is time-independent, we also know that p_0 is constant along its geodesic:

$$p_0 = \text{const} = -E_{\text{em}}\sqrt{1 - \frac{2GM_*}{R_*}} \quad (31)$$

along the entire null geodesic. An observer stationary at infinity has four-velocity $u^\mu \doteq (1, \vec{0})$, and therefore measures

$$E_{\text{obs}} = -p_0 = E_{\text{em}}\sqrt{1 - \frac{2GM_*}{R_*}}, \quad (32)$$

yielding a redshift

$$z_{\text{surf}} = \frac{E_{\text{em}} - E_{\text{obs}}}{E_{\text{obs}}} = \frac{1}{\sqrt{1 - \frac{2GM_*}{R_*}}} - 1 = 0.09. \quad (33)$$

(d) [15 pts] As described in lecture, the mass M_* is *not* what one would get by integrating all fluid density elements over the proper volume of the star’s interior. Let us define M_p as the mass that would be obtained by this integration. Re-integrate the TOV equations for the central density 10^{15} gm/cm^3 , but add the equation

$$\frac{dm_p}{dr} = 4\pi\rho r^2\sqrt{g_{rr}} = 4\pi\rho r^2\left(1 - \frac{2m(r)}{r}\right)^{-1/2}. \quad (34)$$

Compute the mass M_p , which is $m_p(R_*)$.

Solution: We do not need to re-integrate the TOV equations because we used Mathematica and have the results in terms of its `InterpolatingFunction` objects—we will just `NIntegrate` the RHS of the above equation to obtain M_p . See the attached worksheet for the calculation. The result is

$$M_p = 1.08501 \text{ km} = 0.73475 M_\odot, \quad (35)$$

so we see that

$$M_p - M_* = 0.079925 \text{ km} = 0.054123 M_\odot \quad (36)$$

of the mass is accounted for by the gravitational binding energy of the star.

(e) [4 pts] The difference between M_p and M_* reflects the fact that the self-gravity of the star contributes to the star’s mass. These differences can be regarded as the “gravitational binding energy” of the star. Compute

$$\Delta \equiv \frac{M_p - M_*}{M_*}. \quad (37)$$

Δ is the fraction of the star’s mass that is due to this binding energy. What is this fraction for this choice of central density?

Solution: From part (d), we see that

$$\Delta = \frac{M_p - M_*}{M_*} = 0.0795 = 7.95\% \quad (38)$$

3. Stability of a TOV star

By computing a range of TOV models, we can assess whether a star is stable against radial (i.e., purely spherical) perturbations. Detailed analysis shows that a stable star satisfies

$$\frac{dM}{d\rho_c} > 0; \quad (39)$$

an unstable model satisfies

$$\frac{dM}{d\rho_c} < 0. \quad (40)$$

(Notice it is ρ_c that appears in this criterion, not $\rho_{0,c}$.) A detailed explanation and derivation of this stability criterion can be found in *Black holes, white dwarfs, and neutron stars: The physics of compact objects*, by S. L. Shapiro and S. A. Teukolsky, §6.8 - 6.9. An intuitive explanation goes as follows:

Suppose we construct a TOV model, and then squeeze it, decreasing its radius by some small amount δR_* . Clearly, the mean density, and thus the central density, will be augmented, since we are decreasing the volume of the object. If the star resists the squeezing, its mass will also increase: It does work to resist the squeezing, work is energy, and mass and energy are equivalent. On the other hand, if the star does *not* resist the squeezing, its mass will decrease: A small squeeze puts it into an energetically more favorable smaller radius. This clearly leads to a runaway, causing the star to collapse.

(a) [15 pts] Compute a range of TOV models with central densities $\rho_{0,c} = 10^{14}, 10^{15}, 10^{16}, 10^{17}$ and 10^{18} gm/cm³. Make a plot of M_* vs ρ_c for these models. Do you see any evidence for a change from stable to unstable behavior?

Solution: See the attached Mathematica worksheet for the calculations and the plot. We do see evidence for a turnover, around $\log(\rho_c) = -2$.

(b) [15 pts] Zoom in on any unstable region you find and locate, as accurately as possible, the marginally stable star ($dM/d\rho_c = 0$). This star has the maximum possible mass for this equation of state in general relativity. What is this maximum mass? Convert your result from km to solar masses.

Solution: We will generate 100 solutions, between $P_{0,c} = 10^{16}$ g/cm³ and $P_{0,c} = 10^{17}$ g/cm³, fit $dM_*/d\rho_c$ from these solutions using **Interpolation**, and solve for the location at which $dM_*/d\rho_c = 0$. Then we will run a single model at that ρ_c , and determine M_*^{\max} . These operations are performed in the attached Mathematica notebook; the result is

$$M_*^{\max} = 1.17656 \text{ km} = 0.7967 M_{\odot}. \quad (41)$$

Comment on solutions to problems 2 and 3: Sarah worked quite hard to track down numerical inconsistencies, in several cases noting that (for example) an error in mass and radius could be traced to an error in the polytrope parameter K . In a few cases, she setup her mathematica notebook *identical* to a solution that was handed in, but got different numbers. Our best hypothesis in these cases is that people may have neglected to clear mathematica's memory buffers (e.g., by quitting and restarting its kernel) before moving on to a new

problem. In these cases, the results might be “contaminated” by data from previous work on the problem. If you lost some points or disagree with our notebook, you might want to check if you’re in this category.

Second comment on solutions to problems 2 and 3: Many people noted that the masses they got out of this situation were uncomfortably small. That’s a reflection of how inaccurate our equation of state actually is.