8.962 Problem Set 11 Solutions

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1. Formation of a black hole: Oppenheimer-Snyder collapse

   An objection to the Schwarzschild black hole solution described in class is that it is "eternal". In this problem, we will see how a Schwarzschild black hole forms in the collapse of a simple, non-singular physical object.

   We will examine the collapse of a "star" composed of pressureless dust. We take the star to be spherical, of initial radius R_*, of mass M, and composed of an isotropic distribution of pressureless dust. By Birkhoff's theorem, the exterior of this star is simply described by the Schwarzschild metric:

$$ds^2_{r>R_*} = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 - 2GM/r} + r^2 d\Omega^2 \ . \tag{1}$$

   Since the interior is spatially isotropic, it is perfectly described by the Robertson-Walker metric. Since the star clearly must collapse under its own self gravity, we use the closed RW metric to describe the interior:

$$ds^2_{r<R_*} = -d\tau^2 + a^2(\tau) R_0^2 \left(d\chi^2 + \sin^2\chi d\Omega^2\right) \ . \tag{2}$$

   (We use different notation for time on the interior and the exterior: The exterior time t is Schwarzschild time, a convenient label very far away, but perhaps confusing in the strong field; the interior time τ denotes the proper time experienced by an element of dust inside the star.)

   (a) [6 pts] The evolution of the scale factor for a closed RW line element turns out to have a simple, closed form solution. Show that the parametric solution

$$\begin{aligned} a &= \frac{a_{\max}}{2}(1 + \cos\eta) \ , & (3) \\ \tau &= \frac{a_{\max} R_0}{2}(\eta + \sin\eta) \ , & (4) \end{aligned}$$

   with $0 \leq \eta \leq \pi$, solves the Friedmann equations for $k = 1$ assuming ρ is given by pressureless dust matter. Since a always appears multiplied by the lengthscale R_0 in the line element, you may set $a_{\max} = 1$. Find a relationship between the initial density ρ_0 and the lengthscale R_0.

Solution: With $k = 1$, the Friedmann equation reads

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{1}{R_0^2 a^2}. \tag{5}$$

Evaluating this at $a = a_{\max} = 1$ (which is the case at $\eta = 0$ in our parameteric solution), the LHS vanishes, and we have

$$\frac{8\pi G\rho_0}{3} = \frac{1}{R_0^2}, \tag{6}$$

or

$$\rho_0 = \frac{3}{8\pi G R_0^2}. \tag{7}$$

Because we know how ρ scales with a (i.e. $\rho \propto a^{-3}$), we can write the Friedmann equation as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{R_0^2}\left(\frac{1}{a^3} - \frac{1}{a^2}\right). \tag{8}$$

From the parameteric equations, we have

$$\dot{a} = \frac{da/d\eta}{d\tau/d\eta} = -\frac{\tan(\eta/2)}{R_0}, \tag{9}$$

and a simple substitution shows that

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\sec^4(\eta/2)\tan^4(\eta/2)}{R_0^2} = \frac{1}{R_0^2}\left(\frac{1}{a^3} - \frac{1}{a^2}\right). \tag{10}$$

(b) [4 pts] This solution for the interior time coordinate τ is only good up to $\tau = \pi R_0/2$. What happens to the interior solution after that?

Solution: The point $\tau = \pi R_0/2$ corresponds to $\eta = \pi$, but at $\eta = \pi$, we have $a = 0$, so the star has contracted to a singular point. The FRW interior solution has now been replaced with the Schwarzschild solution everywhere; it no longer describes the spacetime of this problem.

We now need to examine the star's surface from the point of view of the external coordinate system.

(c) [6 pts] Consider a purely radial "orbit" (i.e., a trajectory with no angular momentum: $L = 0$). For a given energy per unit mass E, find the radius R at which the radial velocity goes to zero.

Solution: With $L = 0$, the effective potential becomes

$$V_{\text{eff}} = 1 - \frac{2GM}{r}, \tag{11}$$

and the radial velocity goes to zero when

$$E^2 = V_{\rm eff}, \tag{12}$$

or

$$r_0 = \frac{2GM}{1-E^2}. \tag{13}$$

We will use this solution to define the “orbital energy” of a dust element at the surface of the star as it begins to collapse.

(d) [8 pts] Using the radial geodesic equation for the Schwarzschild geometry and the relationship between E and R you found in (b), write down an integral for the proper time τ it takes for a fluid element at the star's surface to fall from its initial radius R_* to r. You should find the answer

$$\tau = -\int_{R_*}^{r} \frac{dr'}{\sqrt{2GM/r' - 2GM/R_*}} \ . \tag{14}$$

(The minus sign is due to the infalling motion of the surface.) By introducing the parameterization

$$r = \frac{R_*}{2}\left(1+\cos\eta\right) \ , \qquad 0 \le \eta \le \pi \ , \tag{15}$$

show that this integral can be evaluated to give

$$\tau \quad = \quad \sqrt{\frac{R_*^3}{8GM}}\left(\eta + \sin\eta\right) \ . \tag{16}$$

Solution: The radial Schwarzschild geodesic equation is

$$\left(\frac{dr}{d\tau}\right)^2 = E^2 - V_{\rm eff} = \frac{2GM}{R_*} - \frac{2GM}{r}, \tag{17}$$

if we have a fluid element which has zero velocity at $r = R_*$ (i.e. which is at rest initially on the surface of the star). Since $\tau = 0$ when $r = R_*$, we have

$$\tau = -\int_{R_*}^{r} \frac{du}{\sqrt{2GM/u - 2GM/R_*}}. \tag{18}$$

(The minus sign is introduced so that $\tau > 0$ as r decreases—the geodesic equation does not fix the sign of $dr/d\tau$.)

If we introduce the paramaterization

$$r = \frac{R_*}{2}\left(1+\cos\eta\right), \tag{19}$$

from which

$$dr = -\frac{R_*}{2}\sin\eta d\eta, \tag{20}$$

then the integral becomes

$$\tau = \frac{R_*^{3/2}}{2\sqrt{2GM}} \int_0^\eta du\ (1+\cos u) = \frac{R_*^{3/2}}{2\sqrt{2GM}} (\eta + \sin\eta)\,, \tag{21}$$

as expected from the problem statement.

We now match the inner and the outer coordinate systems: We require that the star's circumference be the same in both the inner and the outer coordinate systems for all η, and we require that the two expressions for the proper time τ experienced by a fluid element on the star's surface be the same for all η.

(e) [8 pts] By enforcing these two conditions, determine the lengthscale R_0 and the Robertson-Walker radius of the star χ_*.

Solution: The circumference of the star at its maximum expansion is given by

$$C = 2\pi R_* \tag{22}$$

as $r \to R_*$ from above, and

$$C = 2\pi R_0 \sin^2\chi, \tag{23}$$

for $r \to R_*$ from below. Ensuring that these are equal at $r = R_*$ gives the χ coordinate of the surface of the star (for all time, since the surface is a "comoving observer" in the interior FRW spacetime, and therefore has constant χ):

$$\chi_* = \sin^{-1}\sqrt{\frac{R_*}{R_0}}. \tag{24}$$

Ensuring that both expressions for the proper time elapsed for the surface to collapse to the origin (of both coordinate systems) agree gives

$$R_0 = \sqrt{\frac{R_*^3}{2GM}}. \tag{25}$$

This implies that

$$\chi_* = \sin^{-1}\sqrt{\frac{2GM}{R_*}}. \tag{26}$$

For the next part of the problem, assume that the star's initial radius is $R_* = 5GM$.

A Schwarzschild black hole's event horizon is a null surface: It is "generated" by null geodesics whose coordinate locations are $r = 2GM$ for all time. The event horizon of a black hole that forms in collapse is "generated" by the null geodesic that begins at the star's center and reaches the surface just as the surface passes through $r = 2GM$; at that point, by Birkhoff's theorem this horizon "generator" will remain at $r = 2GM$ for all time.

(f) [8 pts] Determine the time τ at which the horizon generator leaves the center of the star.

Hint: It is easiest to solve for this geodesic by noting that the parametric solution for the closed RW spacetime allows us to write

$$ds^2 = a(\eta)^2 R_0^2 \left(-d\eta^2 + d\chi^2 + \sin^2\chi\, d\Omega^2\right) \ . \tag{27}$$

An outward propagating null geodesic thus obeys $d\chi/d\eta = 1$. Using the value of χ you found for the star's surface, it should thus be easy to integrate backwards from the moment that the star's surface crosses $r = 2GM$ to determine the value of η at which the horizon generator leaves the star's center. You then just need to convert to τ.

Solution: Using the hint, and integrating the geodesic equation, we find that

$$\chi_* = \sin^{-1}\sqrt{\frac{2GM}{R_*}} = \Delta\eta \tag{28}$$

where

$$\Delta\eta = \eta_2 - \eta_1, \tag{29}$$

where η_2 is the parameter at which the light ray reaches the surface of the star, and η_1 is the parameter when the light ray leaves the origin.

When $\eta = \eta_2$, we have $r = 2GM$, so

$$2GM = \frac{R_*}{2}\left(1 + \cos\eta\right) = \frac{5GM}{2}\left(1 + \cos\eta_2\right), \tag{30}$$

implying that

$$\cos\eta_2 = -\frac{1}{5}. \tag{31}$$

Then we find that

$$\eta_1 = \eta_2 - \Delta\eta = \cos^{-1}\left(-\frac{1}{5}\right) - \sin^{-1}\sqrt{\frac{2}{5}} \approx 1.08744. \tag{32}$$

At this value of η_1, we have

$$\tau_1 = \sqrt{\frac{125(GM)^3}{8GM}}\left(\eta_1 + \sin\eta_1\right) \approx 7.79846GM. \tag{33}$$

(g) [5 pts] On a spacetime diagram, sketch the evolution of the star's surface and of the event horizon.

Solution: We will plot the surface of the star and the null geodesic which generates the horizon in several different ways. First, consider only the interior coordinates, and plot the η and χ coordinates of the horizon and the surface. The result is Figure 1. The horizon generator rushes out to meet the surface at $\chi = \chi_*$ and $\eta = \eta_2$.

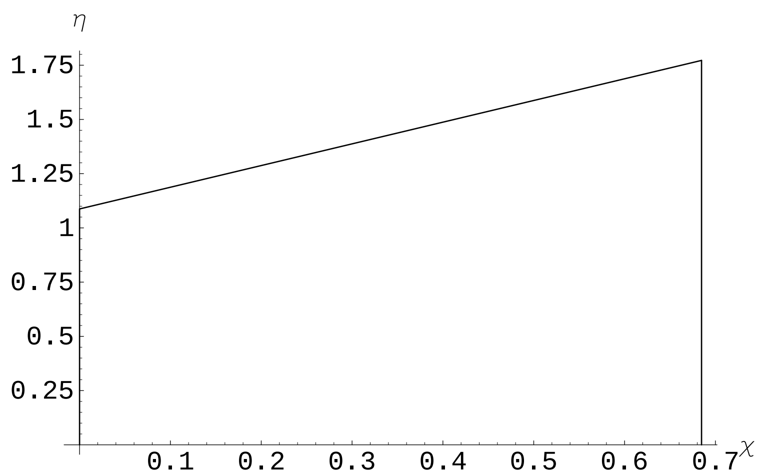

Figure 1: The generator of the horizon and the surface of the star, plotted in the χ, η plane.

Alternately, we could plot the situation in "extended exterior coordinates". Since the exterior coordinate system is invalid in the interior of the star (where the horizon generator originates), we need a sensible way to extend the exterior coordinates into the interior of the star. We will choose an r coordinate inside the star such that the area of constant τ, constant r spheres is $4\pi r^2$. By examining the angular portions of the interior and exterior metrics, we see that this condition implies that

$$r = a(\tau) R_0 \sin\chi \tag{34}$$

inside the star, and that the usual r outside the star satisfies this constraint. In these coordinates, we obtain Figure 2.

2. Consider a static, spherical star cluster in which all stars move in circular orbits. Ignore collisions between stars (i.e., approximate the stars as non-interacting dust). Adopt Schwarzschild-type coordinates with $r = 0$ at the center of the cluster and write the metric in the form

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2 d\Omega^2 \ . \tag{35}$$

(Historical note: A cluster of this type is known as an "Einstein cluster," and was analyzed by Albert Einstein in 1939.)

(a) [6 pts] Find $e^{2\Lambda}$ and $d\Phi/dr$ in terms of $m = \int_0^r 4\pi\rho r^2 dr$, where $\rho = \rho(r)$ is the stars' mass density in the cluster. (We assume that there are enough stars that a continuum treatment is accurate.) Hint: Your final equations should be similar to the TOV equations but with one rather crucial difference.

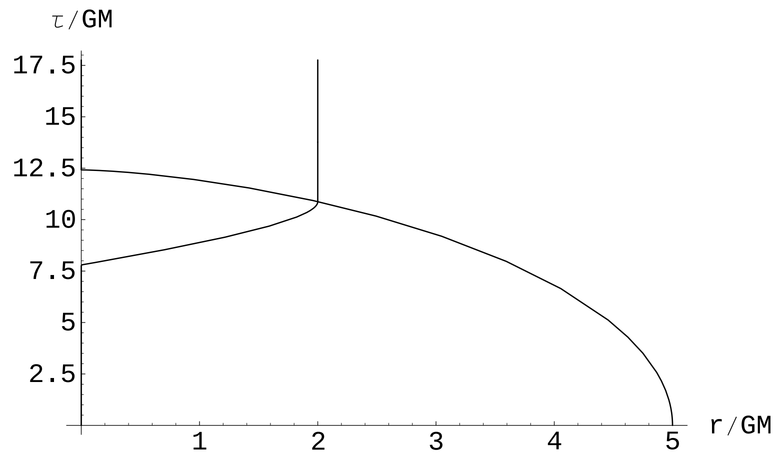

Figure 2: The generator of the horizon and the surface of the star, in the extended r, τ plane.

Hint 1: The cluster can be regarded as a "star", but one in which the restoring force against gravity is orbital motion ("centrifugal force", intuitively) rather than pressure. We assume that there are enough bodies in the cluster that, averaging over their individual motions, the cluster can be regarded as homogeneous, isotropic, and static.

Hint 2: From the circular motion of the bodies that make up the cluster, you should be able to convince yourself that $T_{tt} \neq 0$, but $T_{rr} = 0$. Furthermore, the density ρ is *defined* as $T_{\mu\nu} v^\mu v^\nu$, where $\vec{v}$ is the 4-velocity of a *static* observer. Since we have $v^\mu \doteq (e^{-\Phi}, 0, 0, 0)$, we deduce that $T_{tt} = \rho e^{2\Phi}$.

Hint 3: Re-examine the derivation of the equations describing a static fluid star (Carroll Sec. 5.8; note that Carroll uses α for Φ and β for Λ). Modify that derivation for the current situation and infer $e^{2\Lambda(r)}$ and $d\Phi/dr$.

Solution: The Carroll solution from Section 5.8 goes through unchanged in this situation, except that $p = 0$, so we obtain

$$m(r) = 4\pi \int_0^r \rho\,(r')\, r'^2 dr'. \tag{36}$$

Φ is given by

$$\frac{d\Phi}{dr} = \frac{Gm(r)}{r\,[r - 2Gm(r)]}, \tag{37}$$

and

$$e^{2\Lambda} = \left[1 - \frac{2Gm(r)}{r}\right]^{-1}. \tag{38}$$

We should clarify a bit the assertion that $T_{rr} = 0$. The components $T_{r\mu}$ represent μ momentum flux in the r direction. The only way for stars to carry momentum is by physically moving around in space. But, circular orbits imply that the r-coordinate of any star is fixed, so there is no way to carry any momentum flux in the r-direction.

As soon as we conclude that $T_{rr} \equiv 0$, we can simply copy Carroll's derivation.

(b) [6 pts] Define an appropriate effective potential $V_{\text{eff}}(r)$. Use it to determine the energy per unit mass $\hat{E}$ and angular momentum per unit mass $\hat{L}$ of a star in the cluster. Your answer should be expressed in terms of r, $m(r)$, and $\Phi(r)$. Determine the orbital frequency $\Omega \equiv d\phi/dt = (d\phi/d\tau)/(dt/d\tau)$.

Solution: There are at least two consistent approaches to defining the effective potential. Bear in mind that our goal is to define a function of r which totally characterizes the behavior of $dr/d\tau$; such a function is often called an "effective" potential, even though it might be very different from an actual potential.

We start by noting that the metric is spherically symmetric and static, so we have conserved momenta associated with time and ϕ:

$$\frac{dt}{d\tau} = g^{tt} p_t/m = g^{tt}(-\hat{E}) = \hat{E} e^{-2\Phi} \tag{39}$$

and

$$\frac{d\phi}{d\tau} = g^{\phi\phi} p_\phi/m = g^{\phi\phi} \hat{L} = \frac{\hat{L}}{r^2} \tag{40}$$

along a geodesic, with $\hat{E}$ and $\hat{L}$ constants of the motion.

The only remaining degree of freedom for geodesic motion (we can always use spherical symmetry to set $\theta = \pi/2$) can be found by requiring that

$$g_{\mu\nu} u^\mu u^\nu = -1 = -e^{-2\Phi} \hat{E}^2 + \left[1 - \frac{2Gm(r)}{r}\right]^{-1} \left(\frac{dr}{d\tau}\right)^2 + \frac{\hat{L}^2}{r^2}, \tag{41}$$

which implies

$$\left(\frac{dr}{d\tau}\right)^2 = \left[1 - \frac{2Gm(r)}{r}\right] \left(e^{-2\Phi} \hat{E}^2 - \left[1 + \frac{\hat{L}^2}{r^2}\right]\right). \tag{42}$$

One method to define a useful effective potential is to isolate the constant $\hat{E}^2$, writing this as

$$\left(1 - \frac{2Gm(r)}{r}\right)^{-1} e^{2\Phi} \left(\frac{dr}{d\tau}\right)^2 = \hat{E}^2 - e^{2\Phi} \left[1 + \frac{\hat{L}^2}{r^2}\right]. \tag{43}$$

This doesn't quite put us in the "nice" form $\dot{r}^2 = E - V_{\text{eff}}(r)$ thanks to all the junk in front of $(dr/d\tau)^2$. However, it will still be the case that circular orbits are defined by the right-hand side and the radial derivative of the right-hand side going to zero. (Looking forward to part (c), marginal stability will occur when the second radial derivative of the right-hand side goes to zero.) The effective potential we define with this is then

$$V_{\text{eff}}(r) = e^{2\Phi}\left[1 + \frac{\hat{L}^2}{r^2}\right]. \tag{44}$$

The first condition we apply is $V_{\text{eff}}(r) = \hat{E}^2$ for a circular orbit. This gives us $\hat{E}$ in terms of $\hat{L}$:

$$\hat{E} = e^{\Phi}\sqrt{1 + \frac{\hat{L}^2}{r^2}}. \tag{45}$$

We next require $dV_{\text{eff}}/dr = 0$:

$$\frac{dV_{\text{eff}}}{dr} = -\frac{2e^{2\Phi}\hat{L}^2}{r^3} + 2e^{2\Phi}\left(1 + \frac{\hat{L}^2}{r^2}\frac{d\Phi}{dr}\right). \tag{46}$$

Using our earlier result for $d\Phi/dr$, this becomes

$$\frac{dV_{\text{eff}}}{dr} = \frac{2e^{2\Phi}[Gm(r)(3\hat{L}^2 + r^2) - \hat{L}^2 r]}{r^3[r - 2Gm(r)]}. \tag{47}$$

Equating to zero, we find

$$\hat{L} = r\sqrt{\frac{Gm(r)}{r - 3Gm(r)}}. \tag{48}$$

Note that this is exactly the same as the vanilla Schwarzschild result, replacing M with $m(r)$.

Plugging this into the condition for the energy, we find

$$\hat{E} = e^{\Phi(r)}\sqrt{\frac{r - 2Gm(r)}{r - 3Gm(r)}}. \tag{49}$$

The energy is *not* simply related to the vanilla Schwarzschild result, except in the exterior where $m(r) \to M$, $e^{\Phi} \to \sqrt{1 - 2GM/r}$.

The orbital frequency (measured by an observer at infinity) is given by

$$\Omega = \frac{d\phi/d\tau}{dt/d\tau} = \frac{e^{2\Phi}\hat{L}}{r^2\hat{E}}. \tag{50}$$

Plugging in for $\hat{E}$ and $\hat{L}$,

$$\Omega = \frac{e^{\Phi(r)}}{r}\sqrt{\frac{Gm(r)}{r - 2Gm(r)}}. \tag{51}$$

Note that this reproduces the Schwarzschild result for the exterior.

A second approach is to define our effective potential as just the right-hand side of $(dr/d\tau)^2$:

$$\left(\frac{dr}{d\tau}\right)^2 = \left[1 - \frac{2Gm(r)}{r}\right]\left(e^{-2\Phi}\hat{E}^2 - \left[1 + \frac{\hat{L}^2}{r^2}\right]\right) \equiv V_{\rm eff}(r)\ . \tag{52}$$

This may seem particularly unusual, since our "potential" now depends on the orbital energy $\hat{E}$. Bearing in mind that our goal is just to have a function that characterizes the radial motion, this shouldn't bother us too much. (Indeed, a "potential" of this form is what we must use to characterize radial motion around a Kerr black hole.) Evaluating the derivative and using the result for $d\Phi/dr$, we find

$$\frac{dV_{\rm eff}}{dr} = \frac{1}{r^4}\left[r\hat{L}^2 + G\frac{dm}{dr}\left(r\hat{L}^2 + r^3(1 - e^{-2\Phi}\hat{E}^2)\right) - 2Gm(r)\left(r^2 + 3\hat{L}^2\right)\right]\ . \tag{53}$$

Enforcing $V = 0$, $dV/dr = 0$ leads us to the same solution for $\hat{E}$ and $\hat{L}$ as we found with the other effective potential.

(c) [6 pts] Use $V_{\rm eff}$ to analyze the stability of orbits of individual stars in the cluster. What local condition must $Gm(r)/r$ satisfy if all orbits at r are to be stable?

Solution: For our first potential, stability requires $d^2V_{\rm eff}/dr^2 > 0$; for the second one, we require $d^2V_{\rm eff}/dr^2 < 0$. In either case, we make the transition when the second derivative passes through zero.

For the first potential, evaluating the derivative, substituting for $d\Phi/dr$, and inserting the above solutions for $\hat{E}$ and $\hat{L}$ *after* performing the derivatives (don't insert first, or you'll erroneously differentiate their r dependence!), we find

$$\frac{d^2V_{\rm eff}}{dr^2} = \frac{2e^{2\Phi}G[rm(r) - 6Gm(r)^2 + r^2dm/dr]}{r^2[r^2 - 5Grm(r) + 6G^2m(r)^2]}\ . \tag{54}$$

In the second case, we find

$$\frac{d^2V_{\rm eff}}{dr^2} = -\frac{2e^{2\Phi}G[rm(r) - 6Gm(r)^2 + r^2dm/dr]}{r^3[r - 3Gm(r)]}\ . \tag{55}$$

In both cases, stability requires

$$r > 6Gm(r) - \frac{r^2}{m(r)}\frac{dm}{dr} \tag{56}$$

or

$$\frac{Gm(r)}{r} < \frac{1}{6} + \frac{r}{6m(r)}\frac{dm}{dr}. \tag{57}$$

The local condition is very similar to the Schwarzschild stability condition $r > 6GM$, but is "softened" a bit by the dependence of mass on radius. Note that at the very edge of the cluster, where $m \to M =$ constant, the radial dependence of mass dies, and the "softening" goes away.

(d) [6 pts] Apply the above results to homogeneous cluster of total mass M and radius R. [Homogeneous means $\rho(r) = \text{const}$, so $m(r) = M(r/R)^3$ for $r \leq R$; you will need to use this to solve for $\Phi(r)$ to complete this part of the problem.] Find the maximum value of GM/R if *all* orbits are to be stable.

Solution: For this cluster, $dm/dr = 3m(r)/r$ inside the cluster, and so the local stability criterion becomes

$$\begin{aligned} \frac{Gm(r)}{r} &< \frac{1}{6} + \frac{r}{6m(r)}\frac{3m(r)}{r} \\ &< \frac{1}{6} + \frac{3}{6} \\ &< \frac{2}{3}. \end{aligned} \tag{58}$$

The *local* criterion suggests we can make the cluster quite compact. However, there's a *global* stability criterion, which follows from the fact that $dm/dr \to 0$ just at the edge of the cluster. Applying this, we find that the cluster is only globally stable if

$$\frac{GM}{R} < \frac{1}{6}\ . \tag{59}$$

Note that we don't actually require Φ here; that's needed for the next part. My apologies for any confusion!

(e) [6 pts] Find the cluster with maximal GM/R, compute the redshift of photons emitted from the cluster's surface, and from its center. When quasars were first discovered, their typical redshift was on the order of $z \sim 0.3$. Could a cluster of this type explain this redshift? Today, quasars are measured with redshifts as high as $z \simeq 6.5$. How well does the relativisic cluster hypothesis explain these quasars?

Solution: This is the part for which we need $\Phi(r)$. Plugging in our mass solution, we have

$$\frac{d\Phi}{dr} = -\frac{GMr}{2GMr^2 - R^3}. \tag{60}$$

Integrating from $r = 0$ to some arbitrary point r, we find

$$\begin{aligned} \Phi(r) &= \Phi_c + \frac{1}{4}\left\{\ln\left(\frac{R}{GM}\right)^3 - \ln\left[\left(\frac{R}{GM}\right)^3 - 2\left(\frac{r}{GM}\right)^2\right]\right\} \\ &= \Phi_c + \frac{1}{4}\ln\left[\frac{R^3}{R^3 - 2GMr^2}\right]\ . \end{aligned} \tag{61}$$

The constant Φ_c is the value of Φ at $r = 0$; its value is found by requiring that this solution match to the exterior solution, $e^{2\Phi_{\rm ext}} = (1 - 2GM/r)$, at $r = R$:

$$\Phi_c + \frac{1}{4}\ln\left[\frac{R^3}{R^3 - 2GMr^2}\right] = \frac{1}{2}\ln\left(1 - \frac{2GM}{R}\right) . \tag{62}$$

From this we can compute Φ_c; the answer is simpler if we now specialize to the maximally compact cluster, setting $R = 6GM$. Then,

$$\begin{aligned} \Phi_c &= -\frac{1}{4}\ln\left[\frac{6^3}{6^3 - 2\cdot 6^2}\right] + \frac{1}{2}\ln\left(1 - \frac{1}{3}\right) \\ &= -\frac{1}{4}\ln\left(\frac{3}{2}\right) + \frac{1}{2}\ln\left(\frac{2}{3}\right) \\ &= -\frac{3}{4}\ln\left(\frac{3}{2}\right) . \end{aligned} \tag{63}$$

Turn now to redshift. The redshift is defined as

$$z = \frac{E_{\rm em} - E_\infty}{E_\infty} , \tag{64}$$

the energy is $E = -\vec{p}\cdot\vec{U}$ where $\vec{U}$ is the 4-velocity of a coordinate stationary observer, so that $\vec{U} \doteq (e^{-\Phi(r)}, 0, 0, 0)$ for an observer at r. Putting these together with $\Phi(r) \to 0$ as $r \to \infty$, we have

$$z = e^{-\Phi(r_{\rm emit})} - 1 . \tag{65}$$

For a photon emitted at the surface of a cluster, $\Phi(r_{\rm emit}) = \Phi(r = R) = -(1/2)\ln(3/2)$, and we have

$$z = \sqrt{3/2} - 1 = 0.225. \tag{66}$$

For the photon emitted at the center of the cluster, $\Phi(r_{\rm emit}) = \Phi_c = -(3/4)\ln(3/2)$, and we have

$$z = (3/2)^{3/4} - 1 = 0.355. \tag{67}$$

These values could in principle have accounted for the redshifts that were initially measured from quasars. An early hypothesis for quasars was that the light we measured came from processes at the core of such star clusters. Once quasars with $z > 0.5$ were found, this hypothesis was dead in the water; only cosmological redshift did a decent job explaining those cases. There's not a chance that one could explain modern quasars with $z > 6$ using this mechanism!

3. Periastron precession

In lecture, we showed that the following equations govern the motion of a test body in the Schwarzschild metric:

$$
\begin{aligned}
\left(\frac{dr}{d\tau}\right)^2 &= \hat{E}^2 - V_{\rm eff}(r)\ , \qquad \text{where} \\
V_{\rm eff}(r) &= \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{\hat{L}^2}{r^2}\right)\ ; \\
\frac{d\phi}{d\tau} &= \frac{\hat{L}}{r^2}\ ; \\
\frac{dt}{d\tau} &= \frac{\hat{E}}{1 - 2GM/r}\ .
\end{aligned}
$$

We will now manipulate these equations to calculate the precession angle of a relativistic orbit. A useful tool for this analysis is to reparameterize the radius: Write

$$
r = \frac{p}{1 + e\cos\psi}\ .
$$

This reparameterization *defines* the orbit's eccentricity e. In the Newtonian limit, p is the orbit's semi-latus rectum [related to the semi-major axis by $p = a(1 - e^2)$], and ψ is an angle called the *true anomoly.* As ψ goes from 0 to 2π, r oscillates from $r_{\rm min}$ to $r_{\rm max}$ and back, where

$$
\begin{aligned}
r_{\rm min} &= \frac{p}{1 + e}\ , \\
r_{\rm max} &= \frac{p}{1 - e}\ .
\end{aligned}
$$

(a) [8 pts] The radii $r_{\rm min}$ and $r_{\rm max}$ are turning points: $\dot{r}$ switches sign, passing through zero. From the rule $\dot{r} = 0$ for $r = r_{\rm min,\ max}$, compute $\hat{E}(p, e)$, $\hat{L}(p, e)$. Make sure they reduce to the correct form as $e \to 0$.

Solution: It is easiest to solve for $\hat{E}^2$ and $\hat{L}^2$. Doing so is rather straightforward algebra. First eliminating $\hat{E}^2$ from the system of equations $\dot{r}(r_{\rm min}) = 0$, $\dot{r}(r_{\rm max}) = 0$, we find

$$
\hat{L}^2 = \frac{p^2 GM}{p - GM(3 + e^2)}\ .
$$

Substituting this back in, we find

$$
\hat{E}^2 = \frac{(p - 2GM)^2 - 4e^2G^2M^2}{p[p - GM(3 + e^2)]}\ .
$$

The energy is given by the positive square root of $\hat{E}^2$; for the angular momentum, either sign of the square root can be taken (depending on whether the orbit is "clockwise" or "counterclockwise" according to the ϕ coordinate).

(b) [12 pts] Compute

$$\frac{d\phi}{d\psi} = \frac{d\phi/d\tau}{dr/d\tau}\frac{dr}{d\psi}$$

(The result is rather simple; if your answer is a mess, something has gone awry.)

Solution: First compute $(d\phi/dr)^2 = (d\phi/d\tau)^2/(dr/d\tau)^2$. Substituting in our results for $\hat{E}^2$ and $\hat{L}^2$, using our reparameterization of r, and cleaning up a bit, we should find

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{e^2\sin^2\psi(p - 6GM + 2eGM\cos\psi)GM}{p[p - GM(3 + e^2)]} \ .$$

We also have

$$\left(\frac{d\phi}{d\tau}\right)^2 = \frac{\hat{L}^2}{r^4} = \frac{GM(1 + e\cos\psi)^4}{p - GM(3 + e^2)} \ ;$$

putting these together, we find

$$\left(\frac{d\phi}{dr}\right)^2 = \frac{p(1 + e\cos\psi)^4}{e^2\sin^2\psi(p - 6GM + 2eGM\cos\psi)} \ .$$

Now, multiply by $(dr/d\psi)^2$ and take the (positive) square root to find

$$\frac{d\phi}{d\psi} = \sqrt{\frac{p}{p - 6GM + 2eGM\cos\psi}} \ .$$

(c) [5 pts] Expand your answer in powers of $1/p$ and integrate:

$$\begin{aligned} \Delta\phi &= \phi \text{ accumulated over a full radial orbit} \\ &= \int_0^{2\pi} \frac{d\phi}{d\psi} d\psi \ . \end{aligned}$$

Your answer, to leading order in $1/p$, should take the form

$$\Delta\phi = 2\pi + \delta\phi \ .$$

(Note, Einstein found $\delta\phi = 6\pi GM/a(1 - e^2)$, where a is semi-major axis. The numerical value of this angle precisely matched the historical anomoly in Mercury's orbit precession, a result which left him very excited. Hopefully you reproduce his result!)

Solution: Expanding, we have

$$\frac{d\phi}{d\psi} = 1 + \frac{3GM}{p} - \frac{eGM}{p}\cos\psi \ .$$

Integrating, we find

$$\Delta\phi = 2\pi + \frac{6\pi GM}{p} \ ,$$

from which we identify

$$\delta\phi = \frac{6\pi GM}{p} \equiv \frac{6\pi GM}{a(1-e^2)} \ .$$